REPRESENTING A MEROMORPHIC FUNCTION AS THE QUOTIENT OF TWO ENTIRE FUNCTIONS OF SMALL CHARACTERISTIC

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We announce the following result and sketch the method of proof.

THEOREM. *There exist absolute constants A and B such that if f is any meromorphic function in the complex plane, then there exist entire functions g and h such that* $f=g/h$ *and such that* $T(r,g)\leqq AT(Br,f)$ *and* $T(r,h)\leqq AT(Br,f)$ *for all* $r>0$.

Here $T(r,f)$ is the Nevanlinna characteristic of f evaluated at r.

Rubel and Taylor [1] have obtained such a representation for certain special classes of meromorphic functions. It is shown in [1] that both the arguments and the moduli of the zeros and poles of f play an important role in such a representation; the proof of the above theorem is based on results in [1] together with a new technique for "balancing" the zeros and poles of f.

We now sketch the proof. Let $Z=\{z_n\}$ be the set of poles of f listed according to multiplicity. Without loss of generality we may assume $f(0)\neq\infty$. Let

$$n(t,Z)=\sum_{|z_n|\leq t}1 \quad \text{and} \quad N(r,Z)=\int_0^r \frac{n(t,Z)}{t}\,dt$$

From [1] it is sufficient to show that there exist absolute constants A' and B' and a set $\tilde{Z}=\{\tilde{z}_n\}$ containing Z such that for all $r>0$,

$$N(r,\tilde{Z})\leqq A'T(B'r,f) \tag{1}$$

and such that for all positive integers k and all $s>r\geqq 1$,

$$\left|\frac{1}{k}\sum_{r<|\tilde{z}_n|\leq s}\left(\frac{1}{\tilde{z}_n}\right)^k\right| \leqq \frac{A'T(B'r,f)}{r^k}+\frac{A'T(B's,f)}{s^k}. \tag{2}$$

$\tilde{Z}$ is constructed in the following way. For each integer $N\geqq 1$, we consider those $z_n\in Z$ such that $2^N<|z_n|\leqq 2^{N+1}$ and relabel them simply $z_1, z_2, \cdots, z_{p_N}$ with $z_j=|z_j|e^{i\theta_j}$ and $|z_j|=2^{N+\alpha_j}$ where $0<\alpha_j\leqq 1$ for $1\leqq j\leqq p_N$. For notational convenience we do not indicate the obvious dependence of z_j, θ_j, and α_j on N. We define

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$$g_N(\theta) = -2\sum_{j=1}^{p_N}\left\{\sum_{m=1}^{\infty} 2^{-m(1+\alpha_j)}\exp[im(\theta-\theta_j)]\right\}$$

and

$$f_N(\theta) = \operatorname{Re} g_N(\theta) + 2p_N.$$

It is immediate that

$$0 \leqq f_N(\theta) \leqq 4\{n(2^{N+1}, Z) - n(2^N, Z)\} \tag{3}$$

and that, for each positive integer k,

$$\sum_{j=1}^{p_N}\left(\frac{1}{z_j}\right)^k + \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta 2-k(N-1)} f_N(\theta)\,d\theta = 0. \tag{4}$$

We let

$$L_N = \left[\frac{1}{2\pi}\int_0^{2\pi} f_N(\theta)\,d\theta\right]$$

where [ ] denotes the greatest integer function and choose θ_j' such that

$$\frac{1}{2\pi}\int_0^{\theta_j'} f_N(\theta)\,d\theta = j, \qquad j = 0, 1, \cdots, L_N. \tag{5}$$

The dependence of θ_j' on N is omitted from the notation.

We let $Z_N' = \{2^{N-1}\exp[i\theta_j']: j=0, 1, \cdots, L_N\}$, $Z' = \bigcup_{N\geq 1} Z_N'$, and $\tilde{Z} = Z \cup Z'$. Condition (1) now follows from (3) and (5). It remains to show $\delta_{k,N}$ is small for all positive integers k and N, where

$$\delta_{k,N} = \left|\sum_{z_j' \in Z_N'}\left(\frac{1}{z_j'}\right)^k - \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta 2-k(N-1)} f_N(\theta)\,d\theta\right|.$$

The quantity $\delta_{k,N}$ is small because the sum involved is essentially an approximating Riemann sum for the integral. The fact that $\delta_{k,N}$ is small combined with (4) enables us to conclude (2), finishing the proof. There are lengthy details which we must take up elsewhere.

Reference

1. L. A. Rubel and B. A. Taylor, *A Fourier series method for meromorphic and entire functions*, Bull. Soc. Math. France **96** (1968), 53–96. MR **39** #4399.

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