BOUNDARY VALUES OF HOLOMORPHIC FUNCTIONS

BY ELIAS M. STEIN

Communicated by J. J. Kohn, May 18, 1970

Let $\mathfrak{D}$ be a bounded domain in C^n with smooth boundary. We shall consider the behavior near the boundary of holomorphic functions in $\mathfrak{D}$. Our results are of two kinds: those valid without any further restriction on $\mathfrak{D}$, and those which require that $\mathfrak{D}$ is strictly pseudo-convex. Detailed proofs will appear in [7].

1. Fatou's theorem and H^p spaces. We assume that $\mathfrak{D}$ is a bounded domain with smooth boundary. We first define the appropriate approach to the boundary which extends the usual nontangential approach and takes into account the complex structure of C^n. Let $w \in \partial\mathfrak{D}$, and let ν_w be the unit outward normal at w. For each $\alpha > 0$ consider the approach region $\mathfrak{A}_\alpha(w)$ defined by

$$\mathfrak{A}_\alpha(w) = \{z \in \mathfrak{D}: |(z - w, \nu_w)| < (1 + \alpha)\delta_w(z), \ |z - w|^2 < \alpha\delta_w(z)\}.$$

Here $(z, w) = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$, $|z|^2 = (z, z)$, and $\delta_w(z)$ denotes the minimum of the distances of z from $\partial\mathfrak{D}$ and from z to the tangent hyperplane to $\partial\mathfrak{D}$ at w.

We shall say that F is *admissibly bounded at* w if $\sup_{z \in \mathfrak{A}_\alpha(w)} |F(z)| < \infty$, for some α; F has an *admissible limit at* w, if $\lim_{z \to w,\, z \in \mathfrak{A}_\alpha(w)} F(z)$ exists, for all $\alpha > 0$. On $\partial\mathfrak{D}$ we shall take the measure induced by Lebesgue measure on C^n; we denote it by $m(\cdot)$, or $d\sigma$. The extension of the classical Fatou theorem is as follows.

THEOREM 1. *Suppose F is holomorphic and bounded in $\mathfrak{D}$. Then F has an admissible limit at almost every $w \in \partial\mathfrak{D}$.*

Note. This is stronger than the usual nontangential approach one would obtain using the theory of harmonic functions in R^{2n}. As is to be observed, the admissible approach allows a parabolic tangential approach in directions corresponding to $2n - 2$ real dimensions.

We consider two types of balls on $\partial\mathfrak{D}$. For any $\rho > 0$ and $w \in \partial\mathfrak{D}$,

(1) $B_1(w, \rho) = \{w' \in \partial\mathfrak{D}: |w - w'| < \rho\}$;

(2) $B_2(w, \rho) = \{w' \in \partial\mathfrak{D}: |(w - w', \nu_w)| < \rho, \ |w - w'|^2 < \rho\}$.

Observe that $m(B_1(w, \rho)) \sim c_1\rho^{2n-1}$, and $m(B_2(w, \rho)) \sim c_2\rho^n$ as $\rho \to 0$.

AMS 1969 *subject classifications.* Primary 3217, 3111; Secondary 3220.

Key words and phrases. Fatou's theorem, strictly pseudo-convex domains, maximal functions, H^p spaces, Kähler metric.

THEOREM 3. *Suppose $F \in N$. Then F has admissible limits at almost every $w \in \partial \mathfrak{D}$.*

We consider the following related maximal functions defined for function on $\partial \mathfrak{D}$:

$$f_j^*(w) = \sup_{\rho > 0} \frac{1}{m(B_j(w, \rho))} \int_{B_j(w,\rho)} |f(w')| \, d\sigma(w'), \qquad j = 1, 2.$$

Then the f_j^* satisfy the usual inequalities for maximal functions. (For $j=1$, see e.g. K. T. Smith [4]; for $j=2$, see e.g. Hörmander [1], or Stein [6], and the works cited there.) We define Mf to be the superposition of these two, i.e. $M(f)(w) = (f_1^*)_2^*(w)$. The main step in the proof of Theorem 1 is an argument of harmonic majorization which is essentially contained in the following lemma.

LEMMA. *Suppose u is continuous in $\overline{\mathfrak{D}}$ and pluri-subharmonic in $\mathfrak{D}$. Let f be the restriction of u to $\partial \mathfrak{D}$. Then for each $\alpha > 0$*

$$\sup_{z \in \mathfrak{A}_\alpha(w)} |u(z)| \leqq C_\alpha M f(w). \tag{3}$$

The same argument also allows an extension to H^p spaces. Suppose $\lambda(z)$ is a smooth real-valued function on C^n, so that $\mathfrak{D} = \{z : \lambda(z) < 0\}$, and $|\nabla \lambda(z^0)| > 0$, whenever $\lambda(z^0) = 0$. For sufficiently small ϵ consider the approximating regions $\mathfrak{D}_\epsilon$ defined by $\mathfrak{D}_\epsilon = \{z : \lambda(z) < -\epsilon\}$. If $0 < p < \infty$, and F is holomorphic in $\mathfrak{D}$, we say that $F \in H^p(\mathfrak{D})$ if

$$\sup_{\epsilon > 0} \int_{\partial \mathfrak{D}_\epsilon} |F(z)|^p d\sigma_\epsilon(z) < \infty.$$

$d\sigma_\epsilon$ is the measure on $\partial \mathfrak{D}_\epsilon$ induced by Lebesgue measure in C^n. It can be shown that the property that $F \in H^p(\mathfrak{D})$ is independent of the particular approximating regions $\mathfrak{D}_\epsilon$ defined above, and is thus intrinsic. It is equivalent with the fact that $|F|^p$ has a harmonic majorant in $\mathfrak{D}$. ("Harmonic" is taken in the usual sense in R^{2n}.)

THEOREM 2. *Suppose $F \in H^p(\mathfrak{D})$. Then*

(a) $$\int_{\partial \mathfrak{D}} \sup_{z \in \mathfrak{A}_\alpha(w)} |F(z)|^p d\sigma(w) \leqq A_{p,\alpha} \sup_{\epsilon > 0} \int_{\partial \mathfrak{D}_\epsilon} |F(z)|^p d\sigma_\epsilon(z);$$

(b) *F has an admissible limit at almost every $w \in \partial \mathfrak{D}$.*

There is an analogue also for the Nevanlinna class N. This class is defined as all holomorphic functions F in $\mathfrak{D}$ for which

$$\sup_{\epsilon > 0} \int_{\partial \mathfrak{D}_\epsilon} \log^+ |F(z)| \, d\sigma_\epsilon(z) < \infty.$$

The proof of Theorem 3 requires a modification of estimate (3) of Lemma 1, where M is replaced by a variant which is finite almost everywhere whenever $f \in L^1(d\sigma)$.[1]

2. **Local Fatou theorem and area integral.** From now on we shall assume that in addition $\mathfrak{D}$ is strictly pseudo-convex. We shall introduce a potential theory in $\mathfrak{D}$ which reflects this property in an intimate way. This will be done in terms of a Kähler metric which we now construct in terms of the geometry of $\partial\mathfrak{D}$. For every $z \in \mathfrak{D}$ sufficiently close to $\partial\mathfrak{D}$, let $n(z)$ denote the normal projection of z on $\partial\mathfrak{D}$. Then the mapping $z \to n(z)$ is smooth. For z near $\partial\mathfrak{D}$ we let ν_z denote the (outward) unit normal at $n(z)$. This induces a direct sum decomposition $C^n = N_z \oplus C_z$, where $N_z = \{C\nu_z\}$ and $C_z = (N_z)^\perp$; the orthogonal complement is taken with respect to the usual (complex) inner product $(\cdot, \cdot)$ on C^n. N_z and C_z have complex dimension 1 and $n-1$ respectively.

LEMMA 2. *There exists a Kähler metric* $ds^2 = \sum g_{ij}(z) dz_i d\bar{z}_j$ *defined on* $\mathfrak{D}$ *with the following properties*:

(a) *The* $g_{ij}(z)$ *are smooth on* $\mathfrak{D}$.

(b) $\sum_{i,j} g_{ij}(z)\zeta_i\bar{\zeta}_j \approx (\delta(z))^{-2}|\zeta|^2$, *for* $\zeta \in N_z$.

(c) $\sum_{i,j} g_{ij}(z)\zeta_i\bar{\zeta}_j \approx (\delta(z))^{-1}|\zeta|^2$, *for* $\zeta \in C_z$.

(d) $|\sum_{i,j} g_{ij}(z)\zeta_i\bar{\zeta}'_j| \leqq c(\delta(z))^{-1}|\zeta||\zeta'|$, *for* $\zeta \in C_z$, *and* $\zeta' \in N_z$.

$\delta(z)$ *denotes the distance of* z *from* $\partial\mathfrak{D}$.

One choice of the metric g_{ij} is the one given near the boundary by

$$g_{ij}(z) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j}[\log 1/\delta(z)].$$

With this metric we form the Laplace-Beltrami operator Δ which is given by

$$\Delta = 4 \sum_{i,j} g^{ij} \frac{\partial^2}{\partial \bar{z}_i \partial z_j},$$

where $\{g^{ij}\}$ is the inverse matrix to $\{g_{ij}\}$. This Laplace operator is elliptic in $\mathfrak{D}$ but degenerates at the boundary in a way which takes into account the strict pseudo-convexity of $\partial\mathfrak{D}$. We study the potential theory for the Kähler manifold $\mathfrak{D}$ with the above metric and Laplace operator Δ, by applying Green's theorem in this set up. The following lemma is needed to carry this out.

[1] The argument at this stage was suggested to me by C. L. Fefferman.

LEMMA 3. *For z near the boundary* $|\Delta[(\delta(z))^n]| \leqq c(\delta(z))^{n+1}$.

The thrust of the lemma is that $(\delta(z))^n$ is approximately "harmonic" with respect to Δ. In effect $(\delta(z))^n$ plays the role near the boundary that $\log 1/|z|$ plays near $|z| = 1$ in the case $n = 1$, when $\mathfrak{D}$ is the unit disc.

To state the main result we define the analogue of the area integral. Let $|\nabla F|^2$ denote the square of the norm of the gradient (taken with respect to the metric ds^2 above) for holomorphic F. Thus

$$|\nabla F|^2 = 2 \sum_{i,j} g^{ij} \frac{\overline{\partial f}}{\partial z_i} \frac{\partial f}{\partial z_j}.$$

For any $\alpha > 0$ we define

$$S(F)(w) = \left(\int_{\mathfrak{A}_\alpha(w)} |\nabla F(z)|^2 d\Omega(z) \right)^{1/2}$$

where $d\Omega$ is the element of volume induced by the metric ds^2. In order to see the meaning of the above suppose for simplicity that $w = 0$, and the unit normal ν_w is along the positive y_1 direction, $z_1 = x_1 + iy_1$. Then in $\mathfrak{A}_\alpha(w)$

$$|\nabla F|^2 \approx y_1^2 \left| \frac{\partial F}{\partial z_1} \right|^2 + y_1 \sum_{k=2}^{n} \left| \frac{\partial F}{\partial z_k} \right|^2,$$

and $d\Omega \approx y_1^{-n-1} dz$, where dz denotes Lebesgue measure in $\mathbf{C}^n$.

THEOREM 4. *Suppose F is holomorphic in $\mathfrak{D}$. Then at almost every $w \in \partial\mathfrak{D}$ the following properties are equivalent*:

(a) *F is admissibly bounded at w.*

(b) *F has an admissible limit at w.*

(c) $S(F)(w) < \infty$.

The idea of the proof is to show that almost everywhere (a)$\Rightarrow$(c), and (c)$\Rightarrow$(b). To prove (a)$\Rightarrow$(b) we use the analogue of the argument involving Green's theorem we gave in [5], but now for the potential theory constructed above. To prove (c)$\Rightarrow$(b) we show first that the finiteness of $S(F)$ implies the finiteness of the standard "area integral", thus implying nontangential convergence for almost every point in question. Secondly, condition (c) can also be used as a Tauberian condition, refining nontangential to admissible convergence.

References

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Princeton University, Princeton, New Jersey 08540