# GIRTHS AND FLAT BANACH SPACES

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J. J. Schäffer [3] introduced an interesting parameter for normed linear spaces. It is termed *girth* and is the infimum of the lengths of all centrally symmetric simple closed rectifiable curves which lie in the boundary of the unit ball. More precisely, let $X$ be a Banach space with norm denoted by $\|\cdot\|$ and with $\dim X \geqq 2$. A *curve* in $X$ will be a *rectifiable geometric curve* defined by Busemann [1] as the equivalence class of curves (i.e., continuous functions from a compact interval of real numbers into the space $X$ with the metric given by $\|\cdot\|$) which have the same standard representation in terms of arc-length. Given a curve $c$ we denote its *length* by $\lambda(c)$; and we denote by $\gamma_c(s)$, $0 \leqq s \leqq \lambda(c)$, its *standard representation in terms of arc-length.* We say that $\gamma_c(0)$ and $\gamma_c(\lambda(c))$ are the *initial* and *final points* of $c$. A curve $c$ is *simple* if $\gamma_c$ is injective. Following common usage, a curve $c$ often stands for the common range of its parametrizations, which is a compact subset of $X$. Thus we say, for example, "$x \in c$," or "the linear hull of $c$," or "$c$ lies in a subset $A$ of $X$," etc.

Let $B$ denote the unit ball of $X$ and $S$ the unit sphere. As usual the *inner metric* $\delta$ *of* $S$ is defined for all $x, y \in S$ by

$$\delta(x, y) = \inf\{\lambda(c) : c \text{ a curve lying in } S \text{ having } x \text{ and } y \text{ for its initial and final points}\}.$$

The *girth of* $B$, denoted by $\operatorname{girth}(B)$, is defined by

$$\operatorname{girth}(B) = 2 \inf\{\delta(x, -x) : x \in S\};$$

equivalently, (cf. [3])

$$\operatorname{girth}(B) = \inf\{\lambda(c) : c \text{ a simple closed curve lying in } S,\ c \text{ centrally symmetric}\},$$

where $c$ is said to be *centrally symmetric* if $\gamma_c(s) = \gamma_c(s + \lambda(c)/2)$, for

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$0 \leqq s \leqq \lambda(c)/2$. We say that the girth of $B$ is *achieved by a curve* $c$ if the latter infimum is achieved as a minimum by $c$. If $X$ is finite dimensional then it is shown in [3] that the girth of $B$ is always achieved by some curve. If $X$ is infinite dimensional, the girth of $B$ may or may not be achieved by some curve.

Let $c$ be a closed curve lying in $S$. If $c$ is also centrally symmetric then from $x \in c$ it follows that $-x \in c$. Hence

$$2 = \|x - (-x)\| \leqq (\tfrac{1}{2})\lambda(c).$$

Therefore the girth of $B$ is always greater than or equal to four. It was shown by the present authors in [2] that there exist spaces (e.g. $C[0, 1]$, see also Example 1 below) such that the girth of $B$ is equal to four *and* it is achieved by some curve $c$. This is surprising because it means that $B$ has an "equator" $c$ whose length is twice the diameter of $B$. By virtue of this and Remark 1 below we make the following definition. *The Banach space $X$ is said to be flat if the girth of its unit ball is four and the girth is achieved by a curve $c$.* It is our purpose here to state some interesting geometric properties of flat Banach spaces.

If $X$ is a flat Banach space and $c$ is a curve which achieves the girth, then for each pair of reals $s$ and $h$ we define the *difference quotients* $\Delta_c(s, h)$ by

$$\Delta_c(s, h) = (\gamma_c(s - h) - \gamma_c(s))/h,$$

where the arguments of $\gamma_c$ are taken modulo 4 because $\gamma_c$ is defined on $[0, 4]$. Furthermore for each $s \in [0, 4]$ we define the *chord set* of $c$ at $s$, denoted by $\chi(c, s)$, by

$$\chi(c, s) = \text{closed convex hull } \{\Delta_c(s + i/n, 1/n) : \text{all } n \geqq 2 \text{ and } i = 1, \cdots, 2n\}.$$

The following theorem states some of the geometric properties of $\chi(s, c)$.

THEOREM 1. *Let $X$ be a flat Banach space, and let $c$ be a curve which achieves the girth. Then for each $s$, $0 \leqq s \leqq 4$,*

$$\gamma_c(s) \in \chi(c, s) \subset S.$$

*Moreover, for each $y \in \chi(c, s)$,*

$$\sup_{x \in \chi(c,s)} \|y - x\| = \sup_{x, z \in \chi(c,s)} \|x - z\| = 2,$$

*i.e., each point $y$ of $\chi(c, s)$ is diametral and the diameter of $\chi(c, s)$ is two.*

*Finally, if* $W=$ *closed affine hull of* $\chi(c, s)$, *then* $W-\gamma_c(s)$ *has codimension one in* $U=$ *closed linear hull of* $c$.

REMARK 1. A convex subset of $S$ is naturally thought of as a *flat* area of $S$. If $B$ is strictly convex, a convex subset of $S$ consists of at most one point; and, in general, a convex subset of $S$ has diameter less than or equal to two. This suggests the following interpretation: Each point $\gamma_c(s)$ of $c$ belongs to a flat area $\chi(s, c)$ of the unit sphere which, in the sense of diameter, is the largest possible. If $X=U$, then by virtue of the last part of the theorem we can say that at each point $\gamma_c(s)$ of $c$ the unit sphere looks "locally" like a hyperplane. This justifies the following definition.

*A Banach space* $X$ *is said to be completely flat if* $X$ *is flat and if* $X$ $=$ *closed linear hull of* $c$, *where* $c$ *is a curve which achieves the girth.*

It follows from the paper [2] by the present authors that if $X$ is flat then $\chi(c, s)$ is not weakly compact and hence $X$ is nonreflexive. However, the following theorem shows that this result can be much improved in many cases.

THEOREM 2. *Let* $X$ *be a completely flat Banach space, and let* $c$ *be a curve which achieves the girth. Let* $X^*$ *be the conjugate space of* $X$ *and* $B^*$ *its unit ball. For each* $x\in c$, *let* $x^*$ *be an element of* $B^*$ *such that* $x^*(x)=1$. *If*

$$\text{closed convex hull } \{x^*\}_{x\in c} = B^*, \tag{1}$$

*then* $X$ *is not isomorphic to the conjugate of any Banach space.*

It is not known whether the theorem holds without hypothesis (1). Examples exist to show that (1) is not satisfied for all completely flat Banach spaces. The following theorem is closely related to Theorem 2.

THEOREM 3. *Let* $X$ *be a completely flat Banach space, and let* $c$ *be a curve which achieves the girth. Suppose that relation* (1) *is satisfied. Then each point* $y\in\chi(c, 0)$ *belongs to a curve* $c^y$ *which achieves the girth. Moreover, for each* $s$, $0\leqq s\leqq 4$, $c^y\cap\chi(c, s)$ *consists of precisely one point, and* $\chi(c, s)$ *is a closed bounded convex subset of* $X$ *which contains no extreme points.*

The remaining theorems are concerned with the space $L^1[0, 1]$.

THEOREM 4. $L^1[0, 1]$ *is a completely flat Banach space. Moreover, each* $y\in L^1[0, 1]$, $\|y\|=1$, *lies on a curve* $c^y$ *which achieves the girth. In general this does not define* $c^y$ *uniquely.*

REMARK 2. It is an immediate corollary that the unit ball in $L^1[0, 1]$ contains no extreme points. We can extend this well-known fact, using Theorem 1, as follows. Each point $y \in L^1[0, 1]$ such that $\|y\| = 1$ belongs to a set $\sum(y)$ which has the properties: $\sum(y)$ is closed and convex; $\sum(y)$ belongs to the unit sphere; $\sum(y)$ has diameter two; $\sum(y)$ is diametral; and the affine hull of $\sum(y)$ has codimension one.

In order to include an explicit example of a flat Banach space, we state in Example 1 below the construction of the curve $c^y$ of Theorem 4.

EXAMPLE 1. Let $y \in L^1[0, 1]$ such that $\|y\| = 1$. Define the function $\sigma: [0, 1] \to L^1[0, 1]$ defined by

$$\begin{aligned} (\sigma(s))(t) &= -y(t), \qquad 0 \leqq t \leqq s, \\ &= y(t), \qquad s < t \leqq 1. \end{aligned}$$

Also define $\sigma^*: [0, 2] \to L^1[0, 1]$ by $\sigma^*(s) = \sigma(s)$, $0 \leqq s \leqq 1$, and $\sigma^*(s) = -\sigma(s-1)$, $1 \leqq s \leqq 2$. Then the curve $c^y$, which has $\sigma^*$ as one of its representations, achieves the girth of $L^1[0, 1]$ and $y \in c^y$.

THEOREM 5. *Let $X$ be a completely flat space and $c$ be a curve which achieves the girth. Let $P$ be a linear mapping of $X$ into $L^1[0, 1]$ such that $\|P\| \leqq 1$. Then the following are equivalent*:

*$P$ is an isometry*;

*$X$ and $L^1[0, 1]$ are isometrically isomorphic*; *and*

*$\|Px\| = 1$ for all $x \in c$.*

COROLLARY. *The space $l_1$ is not flat.*

The proofs will be given elsewhere.

## REFERENCES

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