A MEASURABLE MAP WITH ANALYTIC DOMAIN AND METRIZABLE RANGE IS QUOTIENT[1]

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The aim of this note is to prove the statement in the title which is the natural generalization of the classical theorem of N. Lusin for separable metrizable spaces; for historical remarks and classical proof see K. Kuratowski [9, §28].

If P is a topological space we let Baire (P) denote the set P endowed with the σ-algebra of all Baire sets in P. Recall that the collection of Baire sets in P is the smallest σ-algebra of sets such that each real valued continuous function is measurable. A mapping $f: P \to Q$ of topological spaces is called Baire measurable or simply measurable, if $f: \text{Baire}(P) \to \text{Baire}\ (Q)$ is measurable. A mapping $f: P \to Q$ of measurable spaces is called quotient if f is surjective measurable mapping such that $X \subset Q$ is measurable if $f^{-1}[X]$ is measurable. Now we are prepared to state our main result; the reader may also read an interesting corollary in Theorem 9 below.

THEOREM 1. *Let f be a Baire measurable mapping of an analytic topological space A into a metrizable space M. Then the graph ρ of f, and $Q = f[P]$, are analytic, and the mapping $f: A \to Q$ is a measurable quotient mapping.*

It should be remarked that Theorem 1 is highly nontrivial, and that we need the whole machinary of analytic spaces theory for the proof. Recall that a separated space A is called analytic if there exists an upper semicontinuous compact valued (abbreviated to usco-compact) correspondence of the space Σ of irrational numbers onto A. Thus for completely regular spaces the analytic spaces are just the K-analytic spaces introduced by G. Choquet [2], [3]. In this note we will work in the class of all *completely regular spaces*, and the reader familiar with [6] will observe immediately that Theorem 1 holds for analytic spaces as defined in [6] for general topological spaces. For the convenience of the reader we summarize all requisite facts about analytic spaces.

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PROPOSITION. *α. Every analytic space is Lindelöf, and the class of analytic spaces is closed under continuous mappings and countable products.*

β. The collection of all analytic subspaces of any space is closed under the Souslin operation. Thence, every Baire set in an analytic space is analytic.

γ. If X and Y are disjoint analytic subspaces of a space P (completely regular!) then $X \subset B \subset P - Y$ for some Baire set B in P. In particular, if X and $P - X$ are analytic then X is a Baire set in P.

δ. If P is a metrizable uncountable analytic space then there exists a set X in P which is not a Baire set.

ϵ. Every separable metrizable space is a one-to-one continuous image of a closed subspace of the space Σ on irrational numbers.

All statements except δ can be found in [6] or [7]; the reader familiar with older papers by G. Choquet or the author or C. A. Rogers or M. Sion on descriptive theory can prove these results without any difficulty. For a proof of assertion δ, it is enough to know that such P contains a subspace P' homeomorphic to the Cantor set $2^{\aleph_0}$; indeed there are at most $2^{\aleph_0}$ Baire sets in P because P is separable, and $2^{\aleph_0}$ has more than $2^{\aleph_0}$ subsets. A standard reference for δ is [9].

In the proof of Theorem 1 we need the following simple consequence of the author's generalization γ [5] of the Lusin's 1st principle.

THEOREM 2. *If g is a continuous mapping of an analytic space P onto a space Q (completely regular!), then f is a measurable quotient (and by α, Q is analytic).*

PROOF. If $g^{-1}[X]$ is a Baire set in P, then $g^{-1}[X]$ and $P - g^{-1}[X]$ are analytic, hence X and $Q - X$ are analytic by α, and finally X is a Baire set by γ.

PROOF OF THEOREM 1. By Theorem 2 it is enough to show that ρ is analytic. Indeed, the projections of ρ onto A and Q are continuous, hence they are measurable quotient mappings; thus $\{x \to \langle x, fx \rangle\} : A \to \rho$ is a measurable isomorphism, and $\{\langle x, fx \rangle \to fx\} : \rho \to Q$ is a measurable quotient, and finally f is a measurable quotient.

The proof of analycity of ρ will be given in three steps.

A. First assume that M is separable. Obviously we may (and shall) assume that M is completely metrizable. By Proposition ϵ there exists a one-to-one continuous mapping h of a closed subspace F of Σ onto M. We shall think of Σ as the product space N^N where N stands for

the set and also the discrete space of natural numbers. Let S denote the set of all finite sequences of natural numbers, and for each $s \in S$ let Σs denote the set of all σ in Σ which extend s. We shall write $s < \sigma$ if s is a restriction of σ. Thus $\Sigma s = E\{\sigma \mid \sigma \in \Sigma,\ s < \sigma\}$. Clearly $\{\Sigma s \mid s \in S\}$ is a base for open sets in Σ. Now for each s in S put

$$Xs = h[F \cap \Sigma s], \qquad Ys = f^{-1}[Xs] \times Xs.$$

By Theorem 2, each Xs is a Baire set in M, thence $f^{-1}[Xs]$ is a Baire set in A, and finally Ys is a Baire set in $A \times M$ by Proposition β. Since $A \times M$ is analytic (by Proposition α), each Ys is analytic by Proposition β. We shall prove that ρ is the Souslin set determined by $\{Ys \mid s \in S\}$, i.e.

$$\rho = \bigcup \{ \bigcap \{ Ys \mid s < \sigma \} \mid \sigma \in \Sigma \}.$$

It will follow from Proposition β that ρ is analytic. The inclusion $\subset$ is obvious. For the proof of the converse inclusion, choose any point $z = \langle x, y \rangle$ in $A \times M - \rho$. Given any element σ in Σ we shall find an $s < \sigma$ such that $z \notin Ys$. If $\sigma \notin F$, then $F \cap \Sigma s = \varnothing$ for some $s < \sigma$, thence $Xs = \varnothing$, thence $Ys = \varnothing$. Now let $\sigma \in F$. By the continuity of h, if $y \neq h\sigma$ then $y \notin Xs$ for some $s < \sigma$, and if $y = h\sigma$ then $fx \notin Xs$ for some $s < \sigma$ because $y \neq fx$. In both cases $z \notin Ys$. The proof is finished. It should be remarked that using the concept of a Souslin set over a space [8] one can omit the use of Σ.

B. Now assume that Q is separable (M need not be separable). Case A applies to the closure of Q in M, and a simple argument gives the result.

C. It remains the case when Q is not separable. Assume that Q is not separable; we shall derive a contradiction. There exists a closed discrete subspace H of Q of cardinal $\aleph_1$. Since $f^{-1}[H]$ is a Baire set, and hence an analytic space, we may assume that $H = Q$. Take any one-to-one continuous mapping g of Q onto a separable metrizable space K. The composite $h = g \circ f$ is Baire measurable (since g is continuous). Case A applies (with f replaced by h), and hence h is a measurable quotient, and K is analytic. Since h is quotient, g must be quotient, hence a measurable isomorphism. Each subset of Q is a Baire set, hence each subset of K must be a Baire set, but this contradicts Proposition δ. This concludes the proof of Theorem 1.

In conclusion we shall state several corollaries to Theorems 1 and 2 to accent the main consequences; then we give several examples to help the reader to understand the assumptions, and finally we consider the case when A is borelian.

A measurable space P is said to be analytic or compact or metriz-

able or separable if $P = \text{Baire}(Q)$ where Q is a topological space which is, respectively, analytic or compact or metrizable or separable metrizable. It is easy to see that P is separable if and only if the σ-algebra of measurable sets is countably generated. In fact separable spaces are often called countably generated.

THEOREM 3. *Every measurable mapping of an analytic measurable space onto a separable one is a quotient mapping, and the range is analytic.*

THEOREM 4. *If P is metrizable, and if Baire (P) is analytic, then P is analytic.*

THEOREM 5. *Every analytic uncountable topological space contains a nonmeasurable set.*

THEOREM 6. *If an analytic space P measurably maps onto an uncountable separable space, then P contains an analytic subset that is not a Baire set.*

PROOF. Theorems 3 and 4 are particular cases of Theorem 1. For Theorem 5 Proposition δ is needed, and for Theorem 6 one should know that in any separable metrizable analytic space there exists an analytic set which is not a Baire set.

EXAMPLE 1. There exists a measurable one-to-one mapping f of the closed unit interval P onto a compact space Q which is not an isomorphism. E.g., take for Q the set P endowed with the compact topology such that 0 is the only cluster point of Q, and let f be the identity.

EXAMPLE 2. Let R be the space of reals, and let Q be R endowed with the topology having the collection of all left open intervals for an open base. Then Baire $(R) = \text{Baire}(Q)$, Q is hereditarily Lindelöf, and Q is not analytic because $Q \times Q$ is not Lindelöf (not even normal; consider the set of all $\langle x, -x \rangle$).

A space P is called borelian if there exists a disjoint usco-compact correspondence of Σ onto P.

THEOREM 7. *If in Theorem 1 the space A is borelian, then ρ is borelian, and hence Q is absolute Baire if f is one-to-one.*

PROOF. The representation of ρ in part A of the proof of Theorem 1 is a disjoint Souslin representation, and all Y's are borelian. By [6] or [7] ρ is borelian.

By [4] or [7] every borelian space is a one-to-one continuous image of a space Q that is a Baire set in some, hence in each [5], compactification of Q. Moreover, P may be taken to be an N_δ in βQ. By Theorem 2 we get:

THEOREM 8. *The following properties of a measurable space P are equivalent*:

1. $P = Baire\ (Q)$, *where Q is borelian.*

2. $P = Baire\ (Q)$, *where Q is a Baire set in some (and hence in each) compactification of Q.*

3. $P = Baire\ (Q)$ *where Q is the intersection of a sequence of cozero sets in βQ.*

The next theorem shows that every Baire measurable mapping of an analytic (borelian) space into a metrizable space is "analytic (borelian) continuous".

THEOREM 9. *Let A be an analytic (borelian) topological space, and let $\{f_n\}$ be a sequence of Baire measurable mappings of A into metrizable spaces. Then there exists an analytic (borelian) topology on A such that each f_n is continuous and the two topologies are Baire equivalent.*

PROOF. We may replace $\{f_n\}$ by a single mapping f. The new topology is defined by projecting the topology of ρ in Theorem 1 (Theorem 7, respectively) onto A.

In conclusion it should be remarked that we have worked with "Baire" representations of measurable spaces by means of topological spaces. It seems that nothing is known about "Borel" representations. Applications to measure theory will be published elsewhere.

REFERENCES

1. E. Čech, *Topological spaces*, 2nd ed., Publ. House Czech. Acad. Sci., Prague, 1966; English transl., Wiley, New York, 1966. MR **35** #2254.

2. G. Choquet, *Ensembles $\mathcal{K}$-analytiques et $\mathcal{K}$-sousliniens. Cas général et cas métrique*, Ann. Inst. Fourier Grenoble **9** (1959), 75–81. MR **22** #3692a.

3. ———, *Theory of capacities*, Ann Inst. Fourier Grenoble **5** (1953/54), 131–295. MR **18**, 295.

4. Z. Frolík, *A contribution to the descriptive theory of sets and spaces*, Proc. Sympos. General Topology and its Relations to Modern Analysis and Algebra (Prague, 1961) Academic Press, New York and Publ. House Czech. Acad. Sci., Prague, 1962, pp. 157–173. MR **26** #3002.

5. ———, *On bianalytic spaces*, Czechoslovak Math. J. **13** (88) (1963), 561–573. MR **28** #5415.

6. ———, *Analytic and borelian sets in general spaces*, Proc. London Math. Soc. (to appear).

7. ———, *A survey of descriptive theory of sets and spaces*, Czechoslovak Math. J. (to appear).

8. ———, *Abstract analytic and borelian sets*, Bull. Soc. Math. France **97** (1969), 357–368.

9. K. Kuratowski, *Topology*. Vol. I, Academic Press, New York and PWN, Warsaw, 1966. MR **36** #840.

10. N. Lusin, *Ensembles analytique*, Gauthier-Villars, Paris, 1930.

11. C. A. Rogers, *Analytic sets in Hausdorff spaces*, Matematika **11** (1964), 1–8. MR **31** #3993.

12 ———, *Descriptive Borel sets*, Proc. Roy. Soc. Ser. A **286** (1965), 455–478. MR **32** #2536.

13. M. Sion, *On analytic sets in topological spaces*, Trans. Amer. Math. Soc. **96** (1960), 341–354. MR **24** #A1357.

14. ———, *Continuous images of Borel sets*, Proc. Amer. Math. Soc. **12** (1961) 385–391. MR **24** #A1358.

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