FOLIATIONS OF CODIMENSION ONE

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In this note we apply results of [6] to obtain some sufficient conditions for a plane field of codimension one on a manifold to be homotopic to a foliation. This and related questions on foliations are discussed in E. Thomas' survey [5, §4] and, for open manifolds, by A. Haefliger [2]. A. V. Phillips has shown [3] that any field of codimension one on an open manifold is homotopic to a foliation.

Let M be a compact riemannian manifold with boundary. It is convenient to work with the normal line field which corresponds to any plane field of codimension one. A line field is defined by a bundle monomorphism $f:\lambda \hookrightarrow \tau$ where λ is some line bundle over M and τ is the tangent bundle; we say λ embeds in τ. A homotopy of plane fields corresponds to a homotopy of bundle monomorphisms. We *require* $f(\lambda|\partial M)$ to be normal to the boundary and homotopies to be relative to the boundary. In particular, $\lambda|\partial M$ is trivial.

It is unknown which line bundles over M embed as the normal fields of foliations. We can however prove a stable theorem. Let $p:M\times S^1\to M$ be the projection map.

THEOREM 1. *For any line bundle $\lambda\to M$, $p^*\lambda$ embeds as the normal field of a folidation of $M\times S^1$.*

This is in contrast to the situation in higher codimension. The normal bundle σ of a foliation must satisfy Bott's condition that the ring generated by the rational Pontrjagin classes of σ vanishes in dimension $>2\dim\sigma$; and if σ does not satisfy Bott's condition neither does $p^*\sigma$. For codimension 2 for example $p_1(\sigma)^2=0$. If λ is the canonical line bundle over RP^m, then $p^*\gamma$ embeds as the normal field of a foliation of $RP^m\times S^1$ and $w_1(p^*\gamma)^m\neq 0$. This foliation is easily described. There is a map from the solid torus $B^m\times S^1$ onto $RP^m\times S^1$ which is a diffeomorphism on int $B^m\times S^1$ and a double cover from $S^{m-1}\times S^1$ to $RP^{m-1}\times S^1\subset RP^m\times S^1$. The Reeb foliation of $B^m\times S^1$ passes to the desired foliation of $RP^m\times S^1$.

We will need the following known fact.

LEMMA 1. *Let $\lambda\to M$ be a line bundle, s a section transverse to the zero section, $N=s^{-1}$ (zero section), and $i:N\subset M$. Then* $w_1(\lambda)\cap[M]=$

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$i_*[N]$ and $\lambda | N = \nu$ (as bundles) where ν is the normal bundle of N in M. If $\lambda | \partial M$ is trivial we may take $N \subset \operatorname{int} M$.

PROOF OF THEOREM 1. Let s and N be as in the lemma with $N \subset \operatorname{int} M$ and let $\hat{M}$ be M cut along N. The construction of [6 p. 339] gives a foliation of $\hat{M} \times S^1$ with a vector field normal to the foliation and inward normal on $\partial \hat{M} \times S^1$. Gluing along $N \times S^1$ we obtain a foliation of $M \times S^1$ and a section t of the normal line field transverse to the zero section and such that $t^{-1}(\text{zero section}) = N \times S^1$. Note that s gives a section of $p^*\lambda$ with the same properties. By the lemma (applied to $M \times S^1$) $p^*\lambda$ is equivalent to the normal line field of the foliation (since the line bundles are classified by w_1 and hence by $[N \times S^1]$).

The following two results for the case where the normal line field is trivial are essentially contained in [6].

THEOREM 2. *If $f: \epsilon^1 \hookrightarrow \tau(M \times S^1)$, then f is homotopic to the normal field of a foliation.*

THEOREM 3. *If ∂M^3 is a union of tori (or empty), then $f: \epsilon^1 \hookrightarrow \tau M^3$ is homotopic to the normal field of a foliation.*

Let v be a nonvanishing vector field in $f(\epsilon^1)$. If v is inward normal on all boundary components then these results are just Corollary 9.4 and Proposition 10.3 of [6]. Otherwise suppose v is outward normal along $N \subset \partial M^3$. After a homotopy we may assume $v = -\partial/\partial t$ on a collar $N \times [0, 2)$ of N in M. Since N has a nonvanishing vector field, $v | N \times \{1\}$ is homotopic to $+\partial/\partial t | N \times \{1\}$. Thus after a homotopy $-v$ is inward normal on $N \times [0, 1]$, v is inward normal on $M - N \times [0, 1)$, and we are reduced to the previous case. The case of $M \times S^1$ differs only in notation.

If $\lambda \to M$ is any line bundle, there is an embedding $f: \lambda \hookrightarrow \tau M \oplus \epsilon^1$. Identifying $\tau(M \times S^1) = p^*(\tau M \oplus \epsilon^1)$, f corresponds to an embedding $p^*f: p^*\lambda \hookrightarrow \tau(M \times S^1)$. This is the type of embedding that occurs in Theorem 1.

THEOREM 4. *For any $f: \lambda \hookrightarrow \tau M \oplus \epsilon^1$, p^*f is homotopic to the normal field of a foliation.*

LEMMA 2. *If λ^1 and μ^n are bundles over X, there is a one-one correspondence between bundle maps $\lambda \hookrightarrow \mu$ and bundle maps $\epsilon^1 \hookrightarrow \lambda \otimes \mu$ (i.e. nonvanishing sections of $\lambda \otimes \mu$) and also between homotopy classes.*

This is well known and follows by tensoring with λ from the fact that $\lambda \otimes \lambda = \epsilon^1$.

PROOF OF THEOREM 4. Let N and ν be as in Lemma 1; we regard ν

as a line bundle on N with a fixed embedding $\nu \hookrightarrow \tau M | N$. As subbundles of $(\tau M \oplus \epsilon^1) | N \supset \tau M | N$, $\lambda | N$ and ν are homotopic by Lemma 2. Hence $p^*(\lambda | N)$ is homotopic to $p^*\nu$ in $p^*((\tau M \oplus \epsilon^1) | N)$. Under the identification of $p^*(\tau M \oplus \epsilon^1)$ with $\tau(M \times S^1)$, $p^*\nu$ is identified with the normal bundle of $N \times S^1$ in $M \times S^1$. Hence $p^*\lambda$ is homotopic to μ on $M \times S^1$ with $\mu | N \times S^1$ normal to $N \times S^1$. Let $\hat{M}$ be M cut along N; then μ is trivial on $\hat{M} \times S^1$ so Theorem 2 implies the result.

In particular, if $f: \lambda \hookrightarrow \tau M$ then, after crossing with S^1, p^*f is homotopic to the normal field of a foliation. To study more general embeddings of $p^*\lambda$ in $\tau(M \times S^1)$ we need the following result of obstruction theory.

LEMMA 3. *Let M^m be orientable, m odd, $i: N \subset \operatorname{int} M$, $\chi(M-N)=0$, $i_*[N] \neq 0$, and $\lambda \to M$ a line bundle with $w_1(\lambda) \cap [M] = i_*[N]$. Then λ embeds in τM and for any embedding $\lambda | N$ is homotopic to the normal bundle of N in M.*

PROOF. Let $\hat{M}$ be M cut along N. $\chi(\hat{M})=0$ so $\hat{M}$ has a nonvanishing vector field normal along the boundary. This vector field provides a line field on M which is an embedding of λ (see the proof of Theorem 1). There is a corresponding section $f: \epsilon^1 \hookrightarrow \lambda \otimes \tau$. We claim that any section $g: \epsilon^1 | N \hookrightarrow (\lambda \otimes \tau) | N$ which extends over M is homotopic to $f | N$. Homotopy classes of sections of $(\lambda \otimes \tau) | N$ are classified by elements $d(f | N, g) \in H^{m-1}(N; \mathcal{C} | N)$ where $\mathcal{C}$ is the bundle of integer coefficients on M twisted by $w_1(\lambda \otimes \tau): \pi_1(M) \to \operatorname{Aut} \boldsymbol{Z}$, [4, §37]. Since $w_1(\lambda \otimes \tau) = m w_1(\lambda) + w_1(\tau)$, $w_1((\lambda \otimes \tau) | N) = w_1(N)$ and $\mathcal{C} | N$ is the orientation bundle of N. Hence $H^{m-1}(N; \mathcal{C} | N) = \boldsymbol{Z}$. Those classes which extend over M are in the image of i^* in the sequence

$$H^{m-1}(M; \mathcal{C}) \xrightarrow{i^*} H^{m-1}(N; \mathcal{C} | N) \to H^m(M, N; \mathcal{C}) \to H^m(M; \mathcal{C}) \to 0.$$

$H^m(M, N; \mathcal{C}) = \boldsymbol{Z}$ using a tubular neighborhood, excision, and the fact that $\lambda | M - N$ is trivial.

$$\begin{aligned} H^m(M; \mathcal{C}) &= \boldsymbol{Z} \quad \text{if } w_1(\lambda \otimes \tau) = 0 \quad \text{iff } w_1(\lambda) = 0, \\ &= \boldsymbol{Z}_2 \quad \text{if } w_1(\lambda \otimes \tau) \neq 0 \quad \text{iff } w_1(\lambda) \neq 0. \end{aligned}$$

Since $i_*[N] \neq 0$, we see that $i^* = 0$. Thus if g extends, then $g \simeq f | N$, which completes the proof.

THEOREM 5. *If M^m is orientable, m even, and $\lambda \to M$ is a line bundle, then any $f: p^*\lambda \hookrightarrow \tau(M \times S^1)$ is homotopic to a line field normal to a foliation.*

PROOF. We may assume λ is nontrivial. Let N be as in Lemma 1 and apply Lemma 3 to $N\times S^1\subset M\times S^1$ to get $f\simeq g$ with $g(p^*(\lambda|N))$ normal to $N\times S^1$ in $M\times S^1$. Then cut and use Theorem 2.

THEOREM 6. *Let M^3 be an orientable 3-manifold with ∂M^3 a union of tori (or empty). Let $i:N^2\subset \text{int } M^3$ (N^2 not necessarily connected or orientable), and $\chi(N)=0$. If $w_1(\lambda)\cap[M]=i_*[N]$, then any $f:\lambda\hookrightarrow\tau M$ is homotopic to a field normal to a foliation.*

PROOF. Apply Lemma 3 and Theorem 3 as above.

Any line bundle λ on a closed 3-manifold M is classified by $w_1(\lambda)$ or by its Poincaré dual $u\in H_2(M^3;\, \mathbf{Z}_2)$, and u is represented by some surface N. If N is connected and nonorientable, then $N=RP^2 \# \cdots \# RP^2$ (h times) and λ embeds in τM if and only if $w_1(\lambda)^3\neq 0$ [6, 11.4] if and only if h is even [1, p. 88]. In general $\lambda\subset\tau M$ if and only if the sum of the genera of the nonorientable components of N is even. If u is represented by an even number of disjoint embedded RP^2's, then by surgery along arcs connecting them in pairs, u can be represented by disjoint Klein bottles. If u is represented by an embedded S^2, then by adding a small handle u can be represented by $S^1\times S^1$. This gives the following.

COROLLARY 1. *If any element of $H_2(M^3;\, \mathbf{Z}_2)$ can be represented by a disjoint set of embedded surfaces with Euler characteristics $\geqq 0$, then for any $\lambda\rightarrow M$, any $f:\lambda\hookrightarrow\tau$ is homotopic to the normal line field of a foliation.*

Notice that if M_1 and M_2 satisfy the hypothesis of Corollary 1 then so does $M_1\# M_2$.

COROLLARY 2. *The conclusion of Corollary 1 holds for any connected sum of the following manifolds: $S^1\times S^2$, RP^3, $S^1\times S^1\times S^1$, and the lens spaces $L(4k,\, 2k-1)$.*

See [1] for the fact that the Klein bottle embeds in $L(4k,\, 2k-1)$ and also for the statements below.

If M^2 is orientable of genus g, then $M^2\times S^1$ has a line field corresponding to the class represented by $M^2\times\{*\}$ and this class cannot be represented by an orientable surface of genus $<g$ or a nonorientable surface of genus $<2g+2$. The other line bundles on $M^2\times S^1$ can be embedded normal to a foliation by Theorem 5 or 6.

Question 1. Can all line bundles on $M^2\times S^1$, M^2 orientable of genus $\geqq 2$, be embedded as the normal fields of foliations?

For the lens space $L(p,\, q)$, if p is odd there are no nontrivial line bundles. If $p=4k+2$ there are no nontrivial line fields. If $p=4k$ there are nontrivial line fields, but $L(4k,\, 2k-1)$ is the only space in which the Klein bottle embeds. (The torus always bounds.) Thus we also leave unanswered

Question 2. Does the nontrivial line bundle on $L(8, 1)$ occur as the normal field of a foliation?

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