ON THE SUZUKI AND CONWAY GROUPS

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The Suzuki and Conway groups were constructed from the six-dimensional complex representation of a central extension of Z_6 by $PSU_4(3)$ in [2]. In fact, if Q is the rational number field, $\sqrt{-3} = \sqrt{3}i = w - \bar{w}$, $w = (-1+\sqrt{3}i)/2$, and $w^3 = 1$, then the following unitary matrices $M_1, \cdots, M_6$ generate a central extension H of Z_6 by the Suzuki group of order $2^{13}3^7 5^2 7(11)(13)$:

$$M_1 = \frac{1}{\sqrt{(-3)}}\left[\begin{pmatrix} 1 & 1 & 1 \\ 1 & w & \bar{w} \\ 1 & \bar{w} & w \end{pmatrix} \oplus \begin{pmatrix} -1 & -1 & -1 \\ -1 & -\bar{w} & -w \\ -1 & -w & -\bar{w} \end{pmatrix} \oplus \begin{pmatrix} -\bar{w} & -w & -1 \\ -w & -\bar{w} & -1 \\ -1 & -1 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & \bar{w} \\ 1 & \bar{w} & w \end{pmatrix}\right],$$

$$M_2 = \operatorname{diag}(w, w, w, w, w, w, \bar{w}, \bar{w}, \bar{w}, \bar{w}, \bar{w}, \bar{w}).$$

The following denote permutation matrices where our 12 variables are $x_1, \cdots, x_6, x_{1'}, \cdots, x_{6'}$.

$$\begin{aligned} M_3 &= (1 \quad 2 \quad 3)(1' \quad 2' \quad 3')(4' \quad 5' \quad 6'), \\ M_4 &= (4 \quad 5 \quad 6)(1' \quad 2' \quad 3')(4' \quad 6' \quad 5'), \\ M_5 &= (1 \quad 2 \quad 6)(4' \quad 3' \quad 2')(1' \quad 6' \quad 5'), \\ M_6 &= (1 \quad 2 \quad 1')(5 \quad 4' \quad 4)(6 \quad 5' \quad 6'). \end{aligned}$$

A lattice $\mathfrak{L}$ fixed by H can be defined in terms of the following partitions $\{1, \cdots, 6'\} = S_i \cup C(S_i)$, $i = 1, 11$. These partitions are permuted by the above permutation matrices:

$$\begin{aligned} S_1 &= \{1, 2, 3, 4, 5, 6\}, \\ S_2 &= \{1, 2, 3, 1', 2', 3'\}, \\ S_3 &= \{1, 2, 4, 2', 4', 5'\}, \\ S_4 &= \{1, 2, 5, 3', 4', 6'\}, \end{aligned}$$

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\begin{aligned}
S_5 &= \{1, 3, 4, 1', 4', 6'\},\\
S_6 &= \{1, 3, 5, 2', 5', 6'\},\\
S_7 &= \{1, 4, 5, 1', 3', 5'\},\\
S_8 &= \{2, 3, 4, 3', 5', 6'\},\\
S_9 &= \{2, 3, 5, 1', 4', 5'\},\\
S_{10} &= \{2, 4, 5, 1', 2', 6'\},\\
S_{11} &= \{3, 4, 5, 2', 3', 4'\}.
\end{aligned}
$$

Then $\mathfrak{L}$ consists of all vectors $(a_1, \cdots, a_{6'})$ satisfying the following:

I. $a_i \in Z(w)$, all i.

II. $a_i - a_j \in \sqrt{-3}(Z(w))$ all i, j.

III. $\sum_{i\in S} a_i \in 3(Z(w))$ for $S = S_j$ or S_j any j.

IV. $3a_1 + \sum_{i=1}^{6'} a_i \in 3\sqrt{-3}(Z(w))$.

Conway's 24-dimensional group G is generated by $M_i \oplus \overline{M}_i$, $i = 1, \cdots, 6$, where M_i is the complex conjugate of M_i and the permutation matrix N corresponding to the permutation

$$(1\ 2\ \bar{6}')(\bar{2}'\ 6\ 5)(3'\ \bar{1}'\ 4)(\bar{1}\ \bar{2}\ 6')(2'\ \bar{6}\ \bar{5})(3'\ 1'\ \bar{4})$$

where our 24 variables are $x_1, \cdots, x_{6'}, x_{\bar{1}}, \cdots, x_{\bar{6}'}$. Also

$$\mathfrak{L} + \overline{\mathfrak{L}} = \{(x_1, \cdots, x_{6'}, \bar{x}_1, \cdots, \bar{x}_{6'}) \mid (x_1, \cdots, x_{6'}) \in \mathfrak{L}\}$$

is left invariant by G. By Conway's characterization [1] of the Leech lattice we see that $\{(a_1, b_1, \cdots, a_{6'}, b_{6'}) \mid$ for some $(x_1, \cdots, x_{6'}) \in \mathfrak{L}$, $a_i = \operatorname{Re} x_i$, $b_i = \operatorname{Im} x_i$, $i = 1, \cdots, 6'\}$ is the Leech lattice. In fact, when this lattice has its scale shrunk by a factor $(2/9)^{1/2}$, we get a unimodular lattice in which every squared length is an even integer greater than two.

Our construction gives three large subgroups of the 12-dimensional group H. First, M_i, $i = 1, \cdots, 5$ generate $2.3^2.\mathrm{PSU}_4(3)$, where $A.B$ denotes an extension of the group A by the group B. A permutation matrix interchanging S_1 and its complement $C(S_1)$ normalizes this subgroup. Also, M_i, $i = 2, \cdots, 6$ generate K isomorphic to $3^5.M_{11}$ or $3.3^5.M_{11}$. Then $K \oplus \overline{K}$ and N generate the subgroup $3^6.2.M_{12}$ of G where the permutation group N, $M_i \oplus M_i$, $i = 3, \cdots, 6$ is imprimitive on the pairs $\{i, \bar{i}\}$, $i = 1, \cdots, 6'$. If M_{12} denotes the action of $3^6.2.M_{12}$ on the pairs of imprimitivity, then we see that M_{12} contains M_{11} in two different ways, first as the subgroup $M_i \oplus M_i$, $i = 3, \cdots, 6$ and second as the subgroup fixing a pair of imprimitivity.

The third large subgroup of H is the subgroup L fixing a lattice point close to the origin. As H is transitive on the closest lattice points of $\mathcal{L}$ to the origin L can be shown to have the same order as $PSU_5(2)$. Also, L and $PSU_5(2)$ have the subgroup $Z_3 \times PSU_4(2)$ in common. In any event, G is the product of its two subgroups $H \oplus \overline{H}$ (direct sum of matrices) isomorphic to H, and $.2$, the subgroup of G fixing a closest lattice point. Also, H is transitive on the second closest lattice points, and G is the product of $H \oplus \overline{H}$ with $.3$, the subgroup of G fixing the second nearest points to the origin. The intersection of these last two factors is $3^5.PSL(2, 11)$.

Finally, our construction allows us to prove that the simple Suzuki group has outer automorphism group of order two (complex conjugation of the matrix group induces the outer automorphism) and the simple Conway group is complete.

References

1. J. H. Conway, *A group of order* 8,315,553,613,086,720,000, Bull. London Math. Soc. **1** (1969), 79–88.

2. J. H. Lindsey, II, *Linear groups of degree* 6 *and the Hall-Janko group*, Theory of Groups: A Symposium, Benjamin, New York, 1969, pp. 97–100. MR **39** #1538.

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