# $K$-THEORETIC INTERPRETATION OF TAME SYMBOLS ON $k(t)$

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In [3] we introduced a canonical resolution for computing the $K$-theory of [4] and we found a map $\psi: K_2(A) \to \kappa_2^{GL}(A)$ where $K_2(A)$ is the group defined by Milnor [5] and $\kappa_2^{GL}(A)$ is the group of [3]. The map $\psi$ was proved surjective if $A$ is a regular ring. In this announcement we indicated how to compute $\kappa_2^{GL}(k(t))$ for the field $k(t)$ of rational functions in one variable $t$. As a biproduct of this work we have proved

THEOREM 1. *Write $K_2(A[t, t^{-1}]) = K_2(A) \oplus X$. Then if $A$ is regular, $X$ has a homomorphic image $K_1(A)$.*

I should like to thank H. Bass for suggesting that Theorem 1, which was buried in my original announcement, be set off as a main result. Bass has informed me that J. Wagoner also has results on the group $X$.

1. **Generalities.** If $R$ is any ring (without unit) recall the path ring $\Omega R = x(1-x)R[x]$. Clearly $\Omega(R[T]) = (\Omega R)[T]$ if $T$ is a free abelian group or monoid. Also $\kappa_2^{GL}(R) \cong \kappa_1^{GL}(\Omega R)$ [3].

PROPOSITION 1[1]. $\kappa_1^{GL}(R[t]) = \kappa_1^{GL}(R)$ *and* $\kappa_1^{GL}(R[t,t^{-1}]) = \kappa_1^{GL}(R) \oplus \overline{K}_0(R^+)$.

This is an easy consequence of results of [1] and [3].

PROPOSITION 2. *If $A$ is regular, then the composition $\omega$ is a surjective homomorphism.*

$$\begin{array}{ccccc} K_2(A[t,t^{-1}]) & \longrightarrow & \kappa_2^{GL}(A[t,t^{-1}]) & \xrightarrow{\cong} & \kappa_1^{GL}(\Omega A[t,t^{-1}]) \\ & & {\scriptstyle \omega} \searrow & & \downarrow \\ & & & & \overline{K}_0((\Omega A)^+) \end{array}$$

Theorem 1 follows from this proposition using results of [1] and [5].

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[1] ADDED IN PROOF. For the second conclusion of Proposition 1 we require that $R$ be of the form $\Omega^n S$ where $S$ is regular.

Let $R[[t]]$ be the ring of formal power series in $t$ and let $R((t)) = S^{-1}R[[t]]$ where $S=\{1, t, t^2, \cdots\}$.

PROPOSITION 3. *There is a split epimorphism* $K_1(R((t)))\to K_0(R)$.

The map is that defined on p. 74 of [1]. We prove that the module $M(\delta)$ defined there is finitely generated and projective over $R$. The proof of this fact is however necessarily different from that offered in [1].

COROLLARY. *The epimorphism of Proposition* 3 *factors through the quotient* $\kappa_1^{GL}(R((t)))$. *Thus there is a commutative diagram*

$$\begin{array}{ccc} K_2(A((t))) \xrightarrow{\psi} \kappa_2^{GL}(A((t))) & \to & \kappa_1^{GL}(\Omega(A)((t))) \\ \downarrow \omega & & \downarrow \\ \overline{K}_0((\Omega A)^+) & \longleftarrow & \kappa_1^{GL}((\Omega A)((t))) \end{array}$$

PROPOSITION 4. *If $k$ is a field, then there is a canonical isomorphism* $\overline{K}_0((\Omega k)^+)\cong k^*$.

This is established by applying the Mayer-Vietoris sequence [5] to the Cartesian square

$$\begin{array}{ccc} (\Omega k)^+ & \to & (Ek)^+ \\ \downarrow & & \downarrow (x\mapsto 1)^+ \\ Z & \longrightarrow & k^+ \end{array}$$

2. **Tame symbols.** Let $K$ be a field with a discrete valuation $v: K^*\twoheadrightarrow Z$, valuation ring $A$ and residue class field $L$. Then the tame symbol [2] is a Steinberg symbol [3], [5]:

$$U(K)\times U(K) \xrightarrow{(\ ,\ )_v} U(L) = L^*$$

defined as follows. Let $v(\pi)=1$, and let $u$, $u_1\in U(K)$. Write $u=a\pi^i$, $u_1=b\pi^j$ where $a$, $b\in U(A)$. Then

$$(u, u_1)_v = \overline{(-1)^{ij}}\,\overline{a^j/b^i}$$

where "bar" is the residue class in $L$.

If $A$ is a commutative ring, then there is a Steinberg symbol

$$\begin{array}{c} U(A)\times U(A)\to K_2(A) \\ (u, u_1) \longmapsto u * u_1 \end{array}$$

defined by $u*u_1=[h_{12}(u), h_{13}(u_1)]$. [5] and the symbol is natural with respect to ring homomorphisms. By the theorem of Matsumoto

[5], there is a unique homomorphism $S_v: K_2(K) \to L^*$ such that the following diagram commutes

$$\begin{array}{ccc} U(K) \times U(K) & \xrightarrow{\cdot * \cdot} & K_2(K) \\ & {\scriptstyle ( \, , \, )_v} \searrow \quad \swarrow {\scriptstyle S_v} & \\ & L^* & \end{array}$$

In the case of $k((t)) = K$, $A = k[[t]]$, $L = k$ we have

PROPOSITION 5. *If $k$ is a field, the following diagram commutes up to sign*

$$\begin{array}{ccc} U(k((t))) \times U(k((t))) & \xrightarrow{\cdot * \cdot} & K_2(k((t))) \\ \downarrow ( \, , \, )_v & & \downarrow \omega \\ k^* & \xleftarrow{\cong} & \overline{K}_0(\Omega k^+). \end{array}$$

COROLLARY. $( \, , \, )_v$ *factors through* $\kappa_2^{GL}(k((t)))$.

Suppose now that $p$ is an irreducible polynomial in $k[t]$, determining the valuation ring $A_p$ over $k$ in $k(t)$. Complete $A_p$ in the $p$-adic topology to get the valuation ring $\hat{A}$, with residue class field $L = k[t]/(p)$ and field of quotients $\widehat{k(t)}$. By the structure theorem for complete local rings [6], $\hat{A} \cong L[[p]]$ and we have maps $k(t) \to \widehat{k(t)} \cong L((p))$ yielding a commutative diagram

$$\begin{array}{ccc} U(k(t)) \times U(k(t)) & \xrightarrow{( \, , \, )_p} & U(L) \\ \downarrow & \nearrow {\scriptstyle ( \, , \, )_p} & \\ U(L((p))) \times U(L((p))) & & \end{array}$$

This diagram, together with the corollary to Proposition 5 applied to $L((p))$ implies

PROPOSITION 6. *The tame symbol $( \, , \, )_p$ on $k(t)$ determined by the irreducible polynomial $p \in k[t]$ factors through $\kappa_2^{GL}(k((t)))$. That is, there is a commutative diagram*

$$\begin{array}{ccc} U(k(t)) \times U(k(t)) & \to & K_2(k(t)) \\ \downarrow ( \, , \, )_p & & \downarrow \psi \\ L^* & \longrightarrow & \kappa_2^{GL}(k(t)) \end{array}$$

*where $L = k[t]/(p)$.*

As a consequence of Proposition 6 and results of [2] we have

THEOREM 2. *If $k$ is a field, there is a split exact sequence*

$$0 \to K_2^{GL}(k) \to K_2^{GL}(k(t)) \to \oplus U(k[t]/(p)) \to 0$$

*where the sum is taken over all monic irreducible polynomials $p$ in $k[t]$.*

I suspect that an analogue of Proposition 6 is valid for any global field. However, one of the main tools in this work, the results of [**1**], is not available for discrete valuation rings in the unequal characteristic case.

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