# SOME RESULTS ON TOPOLOGICAL NEIGHBOURHOODS

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Communicated by William Browder, December 5, 1969

Full proofs of results announced here are to be found in [**10**]. We consider the problem of classifying germs of neighbourhoods of locally flat topological submanifolds. Let $M^n$ be a fixed topological manifold. An *r-neighbourhood* of $M$ is a pair $(i, N)$ where $N$ is a topological manifold of dimension $n+r$ and $i: M \to N$ a locally flat embedding satisfying $i^{-1}(\partial N) = \partial M$. Denote the set of germs of $r$-neighbourhoods (i.e. equivalence classes under homeomorphism defined near $i(M)$ and fixing $M$) by $\mathfrak{N}_r(M)$.

MAIN THEOREM. *There is a countable* CW *complex* B $\mathbf{Top}_r$ *and a function* $c: \mathfrak{N}_r(M) \to [M, B\ \mathbf{Top}_r]$, *which is a bijection if* $\partial M = \emptyset$ *and* $(n, r) \neq (1, 3)$, *or if* $\partial M \neq \emptyset$ *and* $(n, r) \neq (1, 3)$ *or* $(2, 3)$.

The omitted cases in the theorem are due to the unsolved 4-dimensional annulus conjecture; an extension to any one of these cases is strictly equivalent to the conjecture.

OUTLINE OF PROOF. We rework Haefliger's theory of microbundle pairs [2] in the topological category. The crucial tool is immersion theory. An *r-microbundle pair* is a pair $\epsilon^k \subset \zeta^{k+r}$ of microbundles, where $\epsilon^k$ denotes the trivial bundle of rank $k$ and the inclusion is locally trivial. Two such are *equivalent* (stably) if they are isomorphic after possibly adding further trivial bundles to both elements and the isomorphism is the identity on the trivial sub-bundle. The set of equivalence classes forms a good 'theory'; the group is $\mathbf{Top}_r = \lim_{k\to\infty} \{\mathrm{Top}_{r+k,k}\}$, where $\mathrm{Top}_{r+k,k} = \{$germs of automorphisms of $R^{r+k}$ fixing $\{0\} \times R^k\}$; the classifying space is $B\ \mathbf{Top}_r$. We associate to the neighbourhood $(i, N)$ the pair $\tau_M \oplus \nu \subset i^*(\tau_N) \oplus \nu$, where $\nu$ is a stable inverse to $\tau_M$ and this defines the function $c$. Using immersion theory we show that a germ of neighbourhood is equivalent to a pair $\tau_M \subset \xi^{n+r}$ and then the theorem follows by diagram chasing from:

FIRST STABILITY THEOREM. $\pi_i(\mathrm{Top}_{r+k,k}) \to \pi_i(\mathbf{Top}_r)$ *is an isomorphism provided* $i \leq k$ *and either* $k+r \geq 5$ *or* $r \leq 2$.

This follows from the PL result (Haefliger [2], see also [**11**, III 5.4]) and the analogous result for $\mathrm{Top}_{r+k,k}/\mathrm{PL}_{r+k,k}$ which, using

*AMS* 1969 *subject classifications.* Primary 5701, 5705; Secondary 5730, 5478, 5560.
*Key words and phrases.* Classification of germs of topological neighborhoods of locally flat submanifolds, microbundle pairs, normal block bundles, normal bundles, approximation and triangulation.

immersion theory again, reduces to a statement about straightening handles (in the sense of Kirby-Siebenmann [**7**]) keeping a PL sub-handle fixed. For $r \leqq 2$ this follows from the results of Kirby [**5**] and for $k+r \geqq 5$ we follow the methods of [**7**] appealing to the relative surgery techniques of [**9**]. The relative statement of immersion theory that we need is contained in an appendix to [**10**] written by Armstrong and the first author and follows from the relative local contractibility theorem of Edwards and Kirby [**6**].

APPLICATIONS.

1. *Normal block bundles.* Let $\mathrm{Top}_r^\sim$, the analogue of $\mathrm{PL}_r^\sim$ [**11**], be the group for (closed) topological block bundles. Suppose $M^n \subset Q^{n+r}$ is a locally flat topological pair with $\partial Q \cap M = \partial M$ and that $|K| = M$ is a triangulation (not assumed to be combinational). There is a map $\chi: B\,\mathrm{Top}_r^\sim \to B\,\mathbf{Top}_r$ and, in the dimensions of the main theorem, liftings of $c(\mathrm{id}, Q)$ over $\chi$ correspond to isotopy classes of normal block bundles $\xi^r/K$ on $M$ in $Q$ (in the sense of [**11**, I §4]). We next prove a stability theorem for $\mathrm{Top}_{r+k,k}^\sim/\mathrm{PL}_{r+k,k}^\sim$ by again reducing to relative handle straightening and deduce homotopy properties of $\chi$ using [**11**, III 2.1]. There is an analogous treatment of open and micro block bundles and we obtain:

THEOREM. *There is a normal open, closed or micro block bundle $\xi^r/K$ on $M$ in $Q$ unique up to isotopy in the following cases:*

(1) $r \geqq 5$ *or* $\leqq 2$,

(2) $r=4$ *and both* $M$ *and* $\partial M$ *are* 1-*connected,*

(3) $r=3$ *and both* $M$ *and* $\partial M$ *are* 2-*connected.*

2. *Normal bundles.* Let $M$ and $Q$ be as above and $M$ not necessarily triangulable. We have a similar analysis for normal open and closed disc and microbundles on $M$ in $Q$ (by Kister [**8**] the theories of open disc and microbundles coincide), and using Haefliger and Wall's results [**3**] for the PL case we obtain:

THEOREM. *Suppose $r \geqq 5$; if $n \leqq r$ then $M$ has a normal microbundle in $Q$, if $n<r$ it is unique up to isotopy. Suppose $r \geqq 6$; if $n \leqq r-1$ then $M$ has a normal closed disc bundle in $Q$ and if $n<r-1$ it is unique up to isotopy.*

3. *Approximation and triangulation.* Combining our results with the triangulation theorems of Kirby and Siebenmann [**7**] we obtain obstruction theories analogous to those of Hirsch [**4**] and Haefliger [**1**] for the problems of triangulating a locally flat submanifold of PL manifold and for the problem of approximating a locally flat embedding of PL manifolds by a PL embedding; the coefficients are in $\pi_i(\mathbf{Top}_r/\mathbf{PL}_r)$ and $\pi_i(\mathrm{Top};\, \mathrm{PL},\, \mathbf{Top}_r;\, \mathbf{PL}_r)$ respectively. But $\mathbf{Top}_r/\mathbf{PL}_r$

is contractible if $r \leqq 2$ by Kirby [5] and Wall [12] while if $r \geqq 3$ we have:

SECOND STABILITY THEOREM. *The natural map* $\mathbf{Top}_r/\mathbf{PL}_r \to \mathrm{Top}/\mathrm{PL}$ *is a homotopy equivalence if* $r \geqq 3$.

We deduce:

THEOREM. (1) *Suppose* $i: M^n \to Q^{n+r}$ *is a locally flat embedding of* PL *manifolds and* $i|\partial M$ *is* PL *then if* $r \geqq 3$ *there is an ambient* $\epsilon$-*isotopy of* $Q$ rel $\partial Q$ *carrying* $i$ *to a* PL *embedding. The same is true if* $r \leqq 2$, $n+r \geqq 5$ *and an obstruction in* $H^3(M, \partial M; Z_2)$ *vanishes.*

(2) *Suppose* $i: M^n \to Q^{n+r}$ *is a locally flat embedding and* $Q$ *is a* PL *manifold. Suppose* $i(\partial M)$ *is a* PL *submanifold of* $\partial Q$ *then if* $n \geqq 5$ *and* $r \leqq 5$ *there is an ambient* $\epsilon$-*isotopy of* $Q$ rel $\partial Q$ *carrying* $M$ *to a* PL *submanifold. If* $r \geqq 3$ *the same is true if and only if the given triangulation of* $\partial M$ *extends over* $M$.

From the approximation theorem[1] (part (1) of above) we have:

COROLLARY. *The* PL *and locally flat knot problems coincide in* codim $\geqq 3$ (*i.e. there is a natural bijection between isotopy classes of* PL *and locally flat embeddings of one* PL *manifold in another*).

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[1] The approximation theorem is "well known." Originally announced by Bryant and Seebeck the result has also been proved by Miller. Rushing, Connelly and Cobb.