THE NUMBER OF ROOTS OF $f(x)=a$

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Theorems 1 and 2 below are extensions of some of the results given in [2]. Some of the results used by way of introduction to these theorems and their corollaries may be found in the author's doctoral dissertation written with the helpful advice and encouragement of Robert F. Brown. Detailed proofs will be given elsewhere.

Throughout this paper $f:X\to Y$ will be a map (continuous function) from a path connected topological space X into a path connected topological space Y, and a will be a point in Y. We are interested in the number of roots $x\in X$ to the equation $f(x)=a$. Two such roots x_0 and x_1 are *equivalent* if there is a path C: $[0, 1]\to X$ from x_0 to x_1 such that $[f\circ C]=[a]$. (Here $[f\circ C]$ denotes the fixed-end-point homotopy class containing $f\circ C$ and a is used both to denote the point $a\in Y$ as well as the constant path at $a\in Y$.) This equivalence is indeed an equivalence relation; an equivalence class of roots will be called a *root class*.

Suppose $\{h_t\}$ is a homotopy of maps $h_t:X\to Y$. Then a root x_0 of $h_0(x)=a$ is said to be $\{h_t\}$ related to a root x_1 of $h_1(x)=a$ iff there is a path C in X from x_0 to x_1 such that the path D in Y defined by $D(t)=h_t(C_t)$ is fixed-end-point homotopic to a. A root x_0 of $f(x)=a$ is *essential* iff for any homotopy $\{h_t\}$ beginning at f, there is a root x_1 of $h_1(x)=a$ to which x_0 is $\{h_t\}$ related. If one root in a root class is essential, then they all are and we say that the root class itself is *essential*. The number of essential root classes is called the *Nielsen number* of (f, a) and is denoted by $N(f, a)$. (This is the Δ_2-Nielsen number of [1] and [2].) $N(f, a)$ is clearly a lower bound for the number of solutions of $f(x)=a$. If f' is homotopic to f then $N(f, a)=N(f', a)$.

The order of the cokernel of the fundamental group homomorphism $f_\#:\pi(X, x_0)\to\pi(Y, f(x_0))$ is denoted by $R(f)$; it is independent of the choice of $x_0\in X$ and depends only on the homotopy class of f. There are always at most $R(f)$ root classes of $f(x)=a$ so, in particular, $R(f)\geqq N(f, a)$. Suppose $\{h_t\}$ is a homotopy of maps h_t: $Y\to Y$ such that h_0 is a homeomorphism leaving a fixed, and h_1 is the identity on Y. Define a loop C in Y at a by $C(t)=h_1(a)$. The group of all elements of $\pi(Y, a)$ with a representative of this form will be denoted by $S(Y, a)$. This group contains the Jiang subgroup $T(Y)\subset\pi(Y, a)$ used

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in [2] and elsewhere. For a root x_0 of $f(x)=a$ we use $s(f, x_0)$ to denote the number of right cosets of $f\#(\pi(X, x_0))$ in $\pi(Y, a)$ that have representatives in $S(Y, a)$.

THEOREM 1. *If x_0 is an essential root of $f(x)=a$, then $N(f, a) \geqq s(f, x_0)$.*

Now if $S(Y, a)=\pi(Y, a)$, then clearly $s(f, x_0)=R(f)$ for any root x_0. Hence

COROLLARY 1. *If $S(Y, a)=\pi(Y, a)$ and $f(x)=a$ has at least one essential root, then every root is essential and $N(f, a)=R(f)$.*

When Y is an H-space, we have $T(Y)=\pi(Y, a)$. This fact was used directly in [2] to show that when Y is an H-space and $N(f, a)>0$, then $N(f, a)=R(f)$. Since, however $T(Y) \subset S(Y, a)$, this result also follows from our Corollary 1.

If Y is a manifold, one may show that any path C in Y may be defined by an equation of the form $C(t)=h_t(y_1)$, where $\{h_t\}$ is an isotopy ending at the identity, and $y_1=C(1)$. Hence, if Y is a manifold, then $S(Y, a)=\pi(Y, a)$. Corollary 1 therefore yields

COROLLARY 2. *If Y is a manifold and $f(x)=a$ has at least one essential root, then every root is essential and $N(f, a)=R(f)$.*

We now suppose that X is compact, normal, and locally path connected as well as path connected, and that Y is Hausdorff and semi-locally simply connected as well as path connected. Then for each root class α we may find a closed subset of X having α in its interior, and containing no other roots of $f(x)=a$ other than those in α. For such a set C we may define $f':(C, C-\alpha)\rightarrow(Y, Y-a)$ by $f'(x)=f(x)$, and the inclusion $e:(C, C-\alpha)\subset(X, X-a)$ is an excision. Hence using any homology theory H_* satisfying the Eilenberg and Steenrod axioms we may define a homomorphism $\omega(\alpha):H_*(X)\rightarrow H_*(Y, Y-a)$ by $\omega(\alpha)=f'_*\circ\epsilon_*^{-1}\circ i_*$, where $i:X\subset(X, X-\alpha)$. This homomorphism is independent of the choice of C, except that $\alpha\subset \operatorname{int} C$ and $\alpha=f^{-1}(a)\cap C$. These maps together with f and the inclusion $j: Y\subset(Y, Y-a)$ are diagrammed on the following page.

THEOREM 2. *If $j_*\circ f_*\neq 0$, then $N(f, a)>0$. If $S(Y, a)=\pi(Y, a)$, then $j_*\circ f_*=R(f)\omega(\alpha)$ for any root class α.*

In the case where X and Y are both compact orientable manifolds of the same dimension, n say, we use integer coefficients, choose generators μ of $H_n(X)$ and ν of $H_n(Y)$ and define the *degree* of f as usual by $f_n(\mu)=(\text{degree} f)\nu$. In this case $j_n:H_n(Y)\rightarrow H_n(Y, Y-a)$ is an

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[d, "i"'] & Y \arrow[dd, "j"] \\
(X, X-\alpha) & \\
(C, C-\alpha) \arrow[u, "e"] \arrow[r, "f'"'] & (Y, Y-a)
\end{tikzcd}$$

isomorphism and it is natural to define the *multiplicity* of a root class α by the equation $\omega(\alpha)(\mu) = m(\alpha)j_n(\nu)$. Using this terminology we have

COROLLARY 3. *If X and Y are compact orientable manifolds and degree $f \neq 0$, then $N(f, a) > 0$. In this event there are $R(f)$ root classes, each is essential, each has the same multiplicity, and the multiplicities sum up to the degree of f.*

By using a result of Schirmer's [3], it is possible to show that when X and Y are both compact, orientable, triangulable manifolds of dimension $\geqq 3$, then there is a map g homotopic to f such that $g(x) = a$ has exactly $R(f) = N(f, a)$ roots. In fact the results given here for Y a manifold could have been obtained from general coincidence theory (finding roots of $f(x) = g(x)$ when both f and g may be varied by homotopies) given in [1] and [2] (the "Δ_1" theory). However, the approach via $S(Y, a)$ seems more direct and also gives results when Y is not a manifold.

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