CROSS SECTIONALLY CONNECTED 2-SPHERES ARE TAME

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W. T. Eaton [4] and Norman Hosay [5] have independently shown that a 2-sphere S in E^3 is tame if each horizontal cross section of S is either a simple closed curve or a point. The purpose of this note is to indicate how to extend Hosay's argument to show that S is tame if each horizontal cross section is connected. This answers a question raised by Bing [2].

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The notation used here is as in [5]. Let $E_t = \{(x, y, z) \in E^3 | z = t\}$.

THEOREM. *Let S be a 2-sphere in E^3 such that $S \cap E_t$ is connected (or void) for each t in E^1. Then S is tame.*

Let $J_t = S \cap E_t$. We suppose $\{t | J_t \neq \emptyset\} = [0, 1]$. The first four parts of Hosay's proof are concerned with showing that S is locally tame modulo $J_0 \cup J_1$ by showing that the complementary domains of S are locally simply connected at each point p of $S - (J_0 \cup J_1)$. For a round open ball U containing p he picks a certain map h taking a disk D into $U \cap \mathrm{Cl}(\mathrm{Int}\, S)$ and wishes to construct a map $g: D \rightarrow U - S$ which agrees with h on Bd D.

We first observe that since a separable metric space can contain only countably many mutually disjoint separators which are not irreducible, the set J_t, $0 < t < 1$, is an irreducible separator of S (and hence of E_t) except for at most countably many values of t. Using Cannon's result [3] we know that each set J_t, $0 < t < 1$, is a taming set. We next observe that if $\{J_i\}$ is a countable collection of taming sets on S the techniques of [1] can be used to construct an ϵ-map of $\mathrm{Cl}(\mathrm{Int}\, S)$ into $\mathrm{Cl}(\mathrm{Int}\, S) - \bigcup J_i$. (Proofs of these observations appear in [6].) Thus we may suppose that $h(D) \cap J_t = \emptyset$ unless J_t is an irreducible separator of E_t. This is the key to extending Hosay's argument.

In part (A) of [5] Hosay uses the fact that if $h(A_i^t)$ is a certain continuum in $h(D) \cap E_t$ then any two points of $h(A_i^t) \cap \mathrm{Int}\, S$ can be

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joined by an arc in $E_t \cap U \cap \text{Int } S$. But this can still be done under our weaker hypotheses, since either $h(A_i^t) \subset \text{Int } S$ or else J_t is an irreducible separator of E_t and we can apply the lemma at the end of this paper, letting $E_t \cap \text{Int } S$ be G, $h(A_i^t)$ be C, and $E_t \cap U$ be N.

In part (B) we need to know that each component of $E_t \cap \text{Int } S \cap U$ is simply-connected. But this will always be the case as long as U is chosen small enough so that it does not contain J_t for any t in $[0, 1]$.

The rest of Hosay's proof that Int S is locally simply connected at p can be used without comment.

To show that S is also locally tame at each point of J_0 and J_1 we note first that if J_0, for example, is nondegenerate then it is a taming set [**3**]. Thus S would be locally tame at each point of J_0. If J_0 is a point it is not hard to construct a tame arc piercing S at J_0. (Details are given in [**6**].) Thus S would still be locally tame at J_0. Similarly, S is locally tame at each point of J_1. Thus S is tame.

We are finished when we prove the following lemma used above to enable part (A) of Hosay's construction to be carried out.

LEMMA. *Suppose G is a complementary domain of an irreducible separator of the plane, and suppose C is a compact continuum in* Cl(G). *Then if N is any planar neighborhood of C each pair of points in $C \cap G$ can be joined by an arc in $G \cap N$.*

PROOF. Let p and q be two points of $C \cap G$, and let J be the irreducible separator of the plane.

In general we may suppose that N is the interior of a disk with holes and Bd N is a finite collection of simple closed curves. The lemma follows when we show that Bd $N \cap G$ can not separate p from q in G.

From the unicoherence of the open disk we may conclude that if Bd $N \cap G$ separates p from q in G, then some component of Bd $N \cap G$ separates p from q in G. Suppose such is the case and call this component α. We will arrive at a contradiction.

Since Cl$(\alpha) \subset$ Bd N we know Cl(α) does not separate p from q in the plane, E^2. Since $E^2 - G$ does not separate them either we know from Janiszewski's Theorem that $(E^2 - G) \cap \text{Cl}(\alpha)$ is neither connected nor void. Thus Cl$(\alpha) - \alpha$ contains at least two points, and hence α is the interior of an arc on Bd N, whose endpoints we call x and y.

Let γ be an arc in G from p to q which intersects α in exactly one point, where it pierces α. Pick connected neighborhoods N_x and N_y of x and y respectively which do intersect γ or C. Let G' be a complementary domain of J different from G. Since J is an irreducible separator x and y must be in Cl(G'). Thus G' intersects N_x and N_y. Hence there is a simple closed curve K in $\alpha \cup N_x \cup G' \cup N_y$ which contains $\alpha \cap \gamma$. But now K must separate p from q since γ pierces it, yet the continuum C contains p and q and misses K. Thus the assumption that Bd $N \cap G$ separates p from q in G leads to a contradiction.

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