ON MORSE THEORY AND STATIONARY STATES FOR NONLINEAR WAVE EQUATIONS

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The nonlinear equations to be discussed here can be written in the form

$$(1)\qquad \partial^2 u/\partial t^2 = Lu + N(x, u),$$

$$(1')\qquad i\,\partial u/\partial t = Lu + N(x, u)$$

where L is a second order elliptic formally selfadjoint differential operator acting on complex-valued functions $u(t, x)$ defined on $R^1 \times R^3$, and $N(x, u) = f(x, |u|^2)u$ is a complex-valued function jointly continuous in x and u with $f(x, r) = o(1)$ as $|r| \to \infty$. A complex-valued function $u(t, x)$ is called a stationary state of (1) [or (1′)] if

(a) $u(t, x)$ satisfies (1) [or (1′)] on $R^1 \times R^3$, and

(b) $u(t, x) = v(x)e^{i\lambda t}$ where λ is some real number and $v(x)$ is a smooth real-valued function defined on R^3, tending to 0 exponentially as $|x| \to \infty$ but not identically zero.

In this article we wish to examine the structure and properties of the stationary states of (1) [or (1′)] by combining recent results of Morse theory on Hilbert manifolds with concrete estimates for elliptic differential operators defined on R^3.

1. Statement of basic results. We begin with two affirmative facts concerning the existence of stationary states.

THEOREM 1. *Let* $L = \Delta - p^2$ (p = const.), $f(x, u) = k(|x|)|u|^\sigma$ *with* $0 < \sigma < 4$ *where* $k(|x|)$ *is a bounded positive continuous function uniformly bounded above zero. Then* (1) *and* (1′) *have (for each* $\lambda^2 < p^2$ *in* (1) *and* $\lambda < p^2$ *in* (1′)) *a countably infinite number of stationary states* $v_N(x, \lambda)$, $N = 0, 1, 2, \cdots$. *Each* $v_N(x, \lambda)$ *has precisely* N *nodal domains in* R^3 *and is nonoscillatory outside some fixed sphere of radius* R *(independent of* N*).*

THEOREM 2. *The* $v_N(x, \lambda)$ *of Theorem* 1 *(apart from a constant multiplier) are limits (as* $m \to \infty$*) of spherically symmetric nondegenerate critical points* $v_{Nm}(x, \lambda)$ *of index* N *of the functional* $\int_{B_m} k(|x|)|u|^\sigma u^2$ *on an infinite dimensional Hilbert manifold* $\mathfrak{M}_m$ *where* B_m *is a ball*

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of radius m in R^3 *centered at the origin.* $\mathfrak{M}_m$ *is obtained by intersecting real projective space* $P^\infty(H)$ *(obtained by identifying the antipodal points of the sphere*

$$\int_{B_m} |\nabla u|^2 + |C_\lambda| u^2 = 1, \qquad C_\lambda = \text{const.},$$

of the Hilbert space $H = \dot{W}_{1,2}(B_m)$*) with spherically symmetric functions of the form* $u(x) = v(r)/r$ *where* $v(0) = 0$ *and* $v(|x|) \in \dot{W}_{1,2}(B_m)$.

On the other hand, we have the following results concerning the *nonexistence* of stationary states.

THEOREM 3. *Let* $Lu = \Delta u - p(x)u$ *where* $p(x) = p^2 + o(|x|^{-1})$ *is a nonnegative function continuous outside some bounded domain. Then neither* (1) *nor* (1′) *possesses stationary states for any* $\lambda \geqq p^2$ *in* (1′) *or* $\lambda^2 \geqq p^2$ *in* (1).

THEOREM 4. *In addition to the hypotheses of Theorem* 3, *suppose* $f(x, u) = |u|^\sigma$. *Then if* $\sigma \geqq 4$, *neither* (1) *nor* (1′) *possesses stationary states for any* λ.

2. **Sketch of proofs.** The results (Theorems 3 and 4) concerning the nonexistence of stationary states follow immediately from Kato [1] and the following "invariant integral" for solutions of

$$\Delta v + \partial F(v)/\partial v = 0$$

defined on R^3 and vanishing sufficiently rapidly at ∞

$$6 \int_{R^3} F(v) dx = \int_{R^3} v \frac{\partial F(v)}{\partial v} dx.$$

Theorem 1 follows by noting that the stationary states of (1′) can be obtained (after scaling) as the critical points of the functional $\int_{R^3} k(x)|v|^\sigma v^2$ on the class of functions

$$U = \left\{ v \,\middle|\, \int_{R^3} |\nabla v|^2 + |p^2 - \lambda| v^2 = 1, v \in \dot{W}_{1,2}(R^3) \right\}.$$

These critical points can, in turn, be approximated by replacing R^3 by the ball B_m of sufficiently large radius m. (The argument is similar for (1).) In addition we restrict attention to critical points C_m of the form $v(x) = w(|x|)/|x|$ where $w(r) \in \dot{W}_{1,2}(0, m)$. We then apply the Morse theory of critical points of the functional

$$G(w) = \int_0^m k(r) r^{1-\sigma} (w(r))^{\sigma+1} dr$$

on the infinite dimensional projective space $\tilde{P}^\infty$ defined by identifying antipodal points of the set

$$\left\{w \mid F(w) = \int_0^m \dot{w}^2 + \mid p^2 - \lambda \mid w^2 = 1, w \in \dot{W}_{1,2}(0, m)\right\}.$$

For a suitable sequence $m_n \to \infty$, these critical points are all nondegenerate and the variational problem satisfies the compactness condition: any sequence $w_n \in \tilde{P}^\infty$ with $\{\text{grad } F(w_n)\}$ weakly convergent implies $\{\text{grad } G(w_n)\}$ is strongly convergent (and hence the Palais-Smale condition C for $1/G(w)$). Applying the results of J. Schwartz [2] to this problem, we note $G(w)$ has a sequence of critical points w_{Nm} ($N=0, 1, 2, \cdots$) of index N such that $G(w_{Nm}) > G(w_{N+1,m})$ for all N. This fact together with the classical oscillation theory for second order ordinary differential equations implies

(i) w_{Nm} has precisely N nodal domains in B_m, and

(ii) that the number of these nodal domains does not change when $m \to \infty$.

An important point in the above arguments (especially in showing that $\lim_{m\to\infty} w_{Nm} = w_N$ is a stationary state) is the following a priori bound for critical points C_m.

LEMMA. *For $m \geq M_0$ sufficiently large and for fixed λ, any critical point $v \in C_m$ satisfies*

$$| v(x) | \leq C(e^{-\beta|x|}/|x|), \qquad \text{for } |x| \geq A, \qquad \beta^2 = (p^2 - \lambda),$$

where C and A are constants independent of m.

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