A $P(\phi)$ QUANTUM FIELD THEORY[1]

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Communicated by Peter D. Lax, December 10, 1969[2]

1. **Introduction.** A long-standing problem of quantum field theory is to prove the existence of solutions to the field equations for realistic physical models. The model we consider is that of a self-interacting boson field in two-dimensional space-time with self-interaction given by an arbitrary semibounded polynomial in the field. The hamiltonian supplied by the physics is given formally in terms of the field ϕ as the sum of a free term and an interaction term:

$$H = H_O + H_I \equiv \frac{1}{2}\int :(\phi_x^2 + \phi_t^2 + m^2\phi^2):dx + \int :P(\phi(x)):dx.$$

Here $m>0$ is the bare mass of the boson, $P(y) = b_{2n}y^{2n} + b_{2n-1}y^{2n-1} + \cdots + b_0$ is a polynomial with $b_{2n}>0$, and the colons represent the operation of Wick or normal ordering (defined below).

The corresponding classical field equation is

$$\phi_{tt} - \phi_{xx} + m^2\phi + P'(\phi) = 0 \tag{1}$$

where (classically) ϕ is a real-valued function of x and t. In quantum field theory, ϕ is a distribution in x and t whose values are operators in some Hilbert space; we seek such a solution ϕ of (1).

The natural Hilbert space for noninteracting bosons is (momentum) Fock space, $\mathfrak{F} = \sum_n \oplus \mathfrak{F}^n$. Here $\mathfrak{F}^0 = \boldsymbol{C}$, $\mathfrak{F}^1 = L_2(\boldsymbol{R})$, and $\mathfrak{F}^n$ is the n-fold symmetric tensor product of $\mathfrak{F}^1$. Thus a vector $\Psi \in \mathfrak{F}$ is a sequence of n-particle vectors $\Psi = (\Psi_0, \Psi_1, \cdots)$ where $\Psi_n(p_1, \cdots, p_n)$ is a symmetric function of n momentum variables. The annihilation operator $a(k)$ on $\mathfrak{F}$ maps $\mathfrak{F}^n$ into $\mathfrak{F}^{n-1}$:

$$(a(k)\Psi)_{n-1}(p_1, p_2, \cdots, p_{n-1}) = n^{1/2}\Psi_n(p_1, \cdots, p_{n-1}, k).$$

The formal adjoint of $a(k)$ is

$$(a^*(k)\Psi)_{n+1}(p_1, p_2, \cdots, p_{n+1}) = (n+1)^{1/2}S\delta(p_{n+1} - k)\Psi_n(p_1, \cdots, p_n)$$

AMS Subject Classifications. Primary 8146; Secondary 2846, 4790.

Key Words and Phrases. Boson fields, Fock space, selfadjoint Hamiltonian, removal of cutoffs, resolvent convergence, self-interaction.

[1] The results announced here are contained in the author's doctoral dissertation written at New York University under the guidance of Professor James Glimm. This work was supported by the U. S. atomic Energy Commission, Contract AT(30-1)-1480.

[2] Manuscript first submitted on October 13, 1969.

where S symmetrizes over the variables $p_1, \cdots, p_{n+1}$. $a^*(k)$ is referred to as a creation operator but is in fact only a densely defined bilinear form, say on $D \times D$ where D consists of vectors with a finite number of particles and that are of compact support and continuous in the momentum variables.

We set up the (formal) problem in $\mathfrak{F}$. The field at $t=0$ is

$$\phi(x) = (4\pi)^{-1/2} \int e^{ikx}[a^*(-k) + a(k)]\mu(k)^{-1/2}dk$$

where $\mu(k) = (k^2+m^2)^{1/2}$. The power of the field $\phi^r(x)$ is Wick ordered by placing creators on the left and annihilators on the right

$$:\phi^r(x): = (4\pi)^{-r/2} \sum_{j=0}^{r} \binom{r}{j} \int a^*(-k_1) \cdots a^*(-k_j)a(k_{j+1}) \cdots a(k_r) \times e^{ix(k_1+\cdots+k_r)} \prod_i \mu(k_i)^{-1/2}dk_i.$$

$:\phi^r(x):$ is a densely defined bilinear form on $D \times D$. By a simple calculation with Fourier transforms, the free hamiltonian

$$H_0 = \int \mu(k)a^*(k)a(k)dk.$$

From this form we see that H_0 has meaning as a multiplication operator

$$(H_0\Psi)_n(p_1, \cdots, p_n) = [\mu(p_1) + \cdots + \mu(p_n)]\Psi_n(p_1, \cdots, p_n).$$

Thus we have given meaning to H as a densely defined bilinear form on $\mathfrak{F}$.

Unfortunately $\mathfrak{F}$ is the wrong Hilbert space for bosons that do self-interact. The expression for H on $\mathfrak{F}$ appears to be too singular to lead to a well-defined selfadjoint operator. In fact one version of Haag's Theorem [**12**] states that Fock space is unsatisfactory because it contains no physically acceptable vacuum vector for H. The central mathematical problems then are to choose an appropriate Hilbert space for the problem; to prove that the hamiltonian and the fields are selfadjoint operators on this Hilbert space; to verify the field equations (1); and to establish the properties of the model that are expected from the physics (see [**5**], [**11**]).

2. **The cutoffs.** We approximate the model by a cutoff version for which $\mathfrak{F}$ is a satisfactory Hilbert space. In fact we employ three cutoffs: a space cutoff, a box cutoff, and an ultraviolet cutoff. These cutoffs are subsequently removed.

Let $g(x)$ be a C_0^∞ function which satisfies $0 \leqq g(x) \leqq 1$ and $g(x)=1$ on a large set $(-X, X)$. The spatially cutoff fields are

$$:\phi^r(g): = (4\pi)^{-r/2} \sum_j \binom{r}{j} \int a^*(-k_1) \cdots a^*(-k_j) a(k_{j+1}) \cdots a(k_r) \cdot \hat{g}(k_1 + \cdots + k_r) \prod_i \mu(k_i)^{-1/2} dk_i$$

where $\hat{g}(k) = \int_{-\infty}^{\infty} e^{ikx} g(x) dx$. The spatially cutoff hamiltonian is $H_g = H_0 + H_{I,g}$ where $H_{I,g} = \sum b_r :\phi^r(g)$.

The box cutoff amounts to replacing momentum integrals by sums and arises from placing the system in a box with periodic boundary conditions. The annihilation operators in the box are defined as

$$a_V(k) = \left(\frac{V}{2\pi}\right)^{1/2} \int_{-\pi/V}^{\pi/V} a(k+l) dl$$

where k is in the lattice $\Gamma_V = \{k \mid k = n2\pi/v, n = 0, \pm 1, \pm 2, \cdots\}$. We introduce also the (ultraviolet) cutoff lattice $\Gamma_{K,V} = \{k \mid k \in \Gamma_V, |k| \leqq K\}$. The cutoff free hamiltonian is defined to be

$$H_{O,V} = \int_\mu ([k]_V) a^*(k) a(k) dk$$

where $[k]_V$ is the lattice point closest to k:

$$[k]_V = 1 \in \Gamma_V, \qquad -\frac{\pi}{V} < 1 - k \leqq \frac{\pi}{V} \cdot$$

We approximate the field and its powers by

$$\phi_{K,V}(g) = (2V)^{-1/2} \sum_{k \in \Gamma_{K,V}} [\overset{*}{a_V}(-k) + a_V(k)] \hat{g}_V(k) \mu(k)^{-1/2}$$

and

$$:\phi^r_{K,V}(g): = (2V)^{-r/2} \sum_j \binom{r}{j} \sum_{k_i \in \Gamma_{K,V}} \overset{*}{a_V}(-k_1) \cdots \overset{*}{a_V}(-k_j)\, a_V(k_{j+1}) \cdots a_V(k_r) \hat{g}_V(k_1 + \cdots + k_r) [\mu(k_1) \cdots \mu(k_r)]^{-1/2}$$

where $\hat{g}_V(k) = \int_{-V/2}^{V/2} e^{ikx} g(x) dx$. Finally the fully cutoff hamiltonian is $H_{K,V,g} = H_{O,V} + H_{I,K,V,g}$ where $H_{I,K,V,g} = \sum_r b_r :\phi^r_{K,V}(g):$.

That we have reduced the problem to a less singular one can be seen from the following facts [1], [3], [9]:

(i) $:\phi^r(g):$, $:\phi^r_{K,V}(g):$, $H_{I,g}$, $H_{I,K,V,g}$, H_g and $H_{K,V,g}$ are all densely defined symmetric operators on $\mathfrak{F}$;

(ii) $H_{I,K,V,g}$ is bounded below (whereas $H_{I,g}$ is not);
(iii) $H_{K,V,g}$ is selfadjoint;
(iv) H_g and $H_{K,V,g}$ are bounded below by a constant independent of K and V but dependent on g;
(v) the infimum of the spectrum of $H_{K,V,g}$ is a simple eigenvalue; that is, $H_{K,V,g}$ has a unique (up to phase) vacuum vector in $\mathfrak{F}$.

3. **Main results.** We are able to remove the box and ultraviolet cutoffs in the following sense.

THEOREM 1. *Let K and V approach ∞. For* Re z *sufficiently negative, the resolvents $R_{K,V}(z) = (H_{K,V,g} - z)^{-1}$ converge uniformly on $\mathfrak{F}$ to the resolvent $R(z)$ of a selfadjoint operator T.*

Let $\mathfrak{D}$ be the natural domain of definition for H_g,

$$\mathfrak{D} = D(H_0) \cap D(H_{I,g}).$$

Using T we can prove the selfadjointness of H_g.

THEOREM 2. *$T = (H_g | \mathfrak{D})^-$ so that H_g is essentially selfadjoint on $\mathfrak{D}$.*

OUTLINE OF PROOFS. Full details will appear elsewhere [9]. We exploit an equivalent representation of the problem: there is a positive measure space Q such that $\mathfrak{F}$ is unitarily equivalent to $L_2(Q)$. On an invariant subspace $Q_{K,V}$ of Q, $H_{O,V}$ is the Hermite operator and $H_{I,K,V,g}$ is multiplication by a polynomial. We use the Feynman-Kac formula [6], [8]:

$$(\Phi, e^{-tH_{K,V,g}}\Psi) = \int \Phi(q(0))^- E_{K,V}(q(s))\Psi(q(t))dQ,$$

where the integration takes place over C the space of continuous paths $q(s)$ in Q, dQ is an appropriate probability measure assigned to C, and $E_{K,V}(q(s)) = \exp\left[-\int_0^t H_{I,K,V,g}(q(t'))dt'\right]$. For all $p < \infty$ $H_{I,K,V,g} \in L_p(Q)$ and $E_{K,V}(q(s)) \in L_p(C, dQ)$ with L_p norms bounded independently of K and V. As K and V approach infinity, $H_{I,K,V,g}$ and $E_{K,V}$, are Cauchy sequences in L_p norm.

From the Feynman-Kac formula and the convergence of $E_{K,V}$ we deduce that the semigroups $\exp(-tH_{K,V,g})$ converge uniformly; by the Laplace formula $R_{K,V}(z) = \int_0^\infty e^{tz} \exp(-tH_{K,V,g})dt$, the resolvents converge uniformly (Theorem 1).

Since $\|H_{I,K,V}E_{K,V}\|_2$ is bounded uniformly in K and V it follows that $H_{I,K,V,g}R_{K,V}(z)\Psi$ and $H_{O,V}R_{K,V}(z)\Psi$ are uniformly bounded for $\Psi \in \bigcup_{K,V} L_\infty(Q_{K,V})$. This implies that the core $\mathfrak{C} = R(z)\bigcup_{K,V} L_\infty(Q_{K,V})$ for T is contained in $\mathfrak{D}$ and that $H_g = T$ on $\mathfrak{C}$. Therefore $T \subset (H_g | \mathfrak{D})^-$, a symmetric extension of a selfadjoint operator, and Theorem 2 follows.

4. **Consequences.** Glimm and Jaffe, in a series of papers [2], [3], [4], have carried out a field theory program for the ϕ^4 model with all cutoffs removed. By their methods we can apply Theorems 1 and 2 to remove the space cutoff for the $P(\phi)$ model.

The existence of a unique vacuum vector for $H_{K,V,g}$ together with the uniform convergence of $R_{K,V}$ lead to the existence of a unique vacuum $\Omega_g \in \mathfrak{F}$ for H_g.

By a theorem of Segal [2], [10], the essential selfadjointness of H_g on $\mathfrak{D}$ permits the removal of the space cutoff from the Heisenberg picture dynamics of local algebras; that is, for suitable operators A, $e^{itH_g}Ae^{-itH_g}$ is independent of g provided $g(x)=1$ on a sufficiently large set.

To remove the cutoff completely from the theory we change Hilbert spaces in the following manner. Let $\omega_g(A)=(\Omega_g, A\Omega_g)$ for A in the C^*-algebra $\mathfrak{A}$ of bounded functions of the local field ϕ. As shown in [3], as $g(x)\to 1$ a subsequence of the ω_g (with a slight modification in the definition) converges

$$\omega_{g_n} \to \omega \in \mathfrak{A}^*.$$

According to the Gelfand-Segal construction [7], ω defines an inner product on a new Hilbert space $\mathfrak{F}_{ren}$ where the operators A of $\mathfrak{A}$ are represented by operators A_{ren}. The Heisenberg dynamics in $\mathfrak{F}$ can be unitarily implemented in $\mathfrak{F}_{ren}$ by a one-parameter strongly continuous group of unitary operators $U(t)$. The physical hamiltonian without cutoffs is defined as the generator of this group $U(t)=e^{-itH}$. H is a positive selfadjoint operator with a vacuum vector Ω, $H\Omega=0$.

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