GENERALISED NUCLEAR MAPS IN NORMED LINEAR SPACES

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1. **Preliminary definitions and notations.** Grothendieck [3] and Pietsch [6] present an exhaustive study of nuclear operators and nuclear maps. The notion of a nuclear operator was extended by Persson and Pietsch in a recent paper [5] and they study in detail the p-nuclear and quasi-p-nuclear maps. In this paper we define and study certain linear maps called λ-nuclear and quasi-λ-nuclear maps. Our definition and generalisation here are motivated by the Köthe sequence spaces and their duality theory. For the special case $\lambda = l^1$ we obtain the nuclear operators and for $\lambda = l^p$ we obtain the p-nuclear maps; also, the special case $\lambda = c_0$ yields the ∞-nuclear operators of Persson and Pietsch. Most of the results in this work are motivated by the work of Persson and Pietsch [5] and Köthe sequence spaces.

We shall briefly outline our assumptions. For definitions not stated here see Garling [1], Köthe [4], Ruckle [7], Sargent [9] and Zeller [10]. Let λ be a symmetric sequence space of scalars and λ^* be its Köthe dual. We shall assume that λ is provided with the Mackey topology of the duality $\langle \lambda, \lambda^* \rangle$ and that this topology is provided by a norm p, p itself being an extended seminorm on ω. We assume now that λ is solid and that it is K-symmetric, i.e., for each $x \in \lambda$ and for each permutation π of I^+ we have $x_\pi \in \lambda$ and $p(x) = p(x_\pi)$. λ is also assumed to be a BK space with AK. We remark that our assumptions imply that $\lambda = \omega$ or $\lambda = l^\infty$ or $\lambda \subseteq c_0$. The space λ^* is now considered as the topological dual of λ and equipped with its natural norm topology.

We pause now to point out that in addition to the spaces l^p, $1 \leq p < \infty$, the sequence spaces $n(\phi)$ of Sargent [8] and the sequence spaces $\mu_{a,p}$ and $\nu_{a,p}$ of Garling [2] serve as examples of the type of sequence spaces λ we consider. Garling shows also that his spaces $\mu_{a,p}$ are in general not linearly homeomorphic to l^p.

Next let E and F be normed linear spaces. Then $\lambda(E)$ is the (vector sequence) space of all vectors $x = (x_n)$, $x_n \in E$ for each n and such that the sequence $(\langle x_n, a \rangle) \in \lambda$ for each $a \in E'$. Formally define

$$\epsilon_\lambda(x) = \sup_{\|a\| \leq 1} p(|\langle x_n, a \rangle|),$$

where p is the norm on λ.

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$\lambda[E]$ is the space of sequences $x=(x_n)$, $x_n\in E$ for each n and such that $(\|x_n\|)\in\lambda$; the space $\lambda[E]$ is equipped with a natural norm topology given by $\|x\|_\pi=p[(\|x_n\|)]$.

2. **λ-nuclear maps.** Let T be a linear map on the normed space E into another, F. We define T to be a λ-nuclear map if T admits the representation

$$Tx = \sum_{n=1}^{\infty} \langle x, a_n\rangle y_n, \qquad x \in E, \tag{1}$$

where $a=(a_n)\in\lambda[E']$ and $y=(y_n)\in\lambda^*(F)$ with $\epsilon_{\lambda^*}(y)<\infty$. There may be other representations of T in the above form. Keeping this in mind, we define

$$N_\lambda(T) = \inf\{\|a\|_\pi\cdot\epsilon_{\lambda^*}(y)\} \tag{2}$$

where the infimum is taken over all possible representations of T in the above form.

We observe that λ-nuclear maps can be defined in the following equivalent way: say T is λ-nuclear if T has the representation

$$Tx = \sum_{n=1}^{\infty} \alpha_n\langle x, u_n\rangle y_n, \tag{3}$$

where $\|u_n\|\leqq 1$ for each n, $\alpha=(\alpha_n)\in\lambda$ and $y=(y_n)\in\lambda^*(F)$ with $\epsilon_{\lambda^*}(y)\leqq 1$. In this case

$$N_\lambda(T) = \inf p(\alpha). \tag{4}$$

Let $N_\lambda(E, F)$ denote the set of all λ-nuclear maps on E into F.

THEOREM 1. *Each λ-nuclear map T is continuous and $\|T\|\leqq N_\lambda(T)$.*

THEOREM 2. *$N_\lambda(E, F)$ is a quasi-normed linear space under the norm N_λ; also if F is a Banach space $N_\lambda(E, F)$ is complete if λ is made of all sequences $u\in\omega$ for which $p(u)<\infty$.*

THEOREM 3. *If $A(E, F)$ denotes the space of all operators T on E which have finite dimensional ranges in F, then $A(E, F)$ is a dense subspace of $N_\lambda(E, F)$.*

COROLLARY. *If F is a Banach space then each $T\in N_\lambda(E, F)$ is a compact linear map and each such T has a separable range space.*

The next two theorems play an important role in the factorization theorem (Theorem 6) characterizing λ-nuclear maps.

THEOREM 4. *Let E, F and G be normed linear spaces. Then we have the following*:

(a) *If* $T \in N_\lambda(E, F)$ *and* $S \in L(F, G)$ *then* $S \circ T \in N_\lambda(E, G)$ *and* $N_\lambda(S \circ T) \leqq \|S\| \cdot N_\lambda(T)$.

(b) *If* $T \in L(E, F)$ *and* $S \in N_\lambda(F, G)$ *then* $S \circ T \in N_\lambda(E, G)$ *and* $N_\lambda(S \circ T) \leqq N_\lambda(S) \cdot \|T\|$.

THEOREM 5. *Let* $\delta = (\delta_n)$ *be a fixed member of* λ. *Then the map* $D: l^\infty \to \lambda$ *defined by* $D(u) = (u_i \delta_i)$ *is a* λ*-nuclear map and* $N_\lambda(D) = p(\delta)$.

THEOREM 6. *Suppose F is a Banach space. Then the map* $T \in L(E, F)$ *is* λ*-nuclear if and only if it can be factorized as follows*:

$$T = Q \circ D \circ P, \qquad E \xrightarrow{P} l^\infty \xrightarrow{D} \lambda \xrightarrow{Q} F$$

where P and Q are continuous linear maps with $\|P\| \leqq 1$ *and* $\|Q\| \leqq 1$ *and D is as defined in Theorem* 5.

3. **Quasi-λ-nuclear maps.** A linear map T on E into F is defined to be quasi-λ-nuclear if there exists a sequence $a = (a_n)$ of elements of E' such that $a \in \lambda[E']$ and $\|Tx\| \leqq p[(|\langle x, a_n\rangle|)]$ for each $x \in E$. Set $Q_\lambda(T) = \inf \|a\|_\pi$, where the infimum is taken over all admissible a. Then one can prove that $Q_\lambda(E, F) \subset L(E, F)$ with $\|T\| \leqq Q_\lambda(T)$. Also $N_\lambda(E, F) \subset Q_\lambda(E, F)$ with $Q_\lambda(T) \leqq N_\lambda(T)$ for $T \in N_\lambda(E, F)$. In the opposite direction we have the following result.

THEOREM 7. *If the Banach space F has the extension property and if* $T \in Q_\lambda(E, F)$ *then* $T \in N_\lambda(E, F)$ *and* $Q_\lambda(T) = N_\lambda(T)$.

We remark also that the above result is true for any pair E, F provided the sequence space λ is complemented. Thus for $\lambda = l^2$ when one gets the quasi-2-nuclear maps and the 2-nuclear maps, we have the (known) result that $N_2(E, F) = Q_2(E, F)$.

4. **λ-nuclear maps and absolutely λ-summing maps.** The linear map T on E into F is said to be absolutely λ-summing if for each $x = (x_n) \in \lambda(E)$, the sequence $Tx = (Tx_n) \in \lambda[F]$. Let now $\lambda = \{x \in \omega : p(x) < \infty\}$.

THEOREM 8. *The linear map T on E into F is absolutely* λ*-summing if and only if there exists a* $\rho > 0$ *such that for each finite system of vectors* $x_1, x_2, \cdots, x_k$ *in E the following inequality holds*:

$$\|(Tx_1, Tx_2, \cdots, Tx_k, 0, 0, \cdots)\|_\pi \leqq \rho \cdot \epsilon_\lambda(x_1, x_2, \cdots, x_k, 0, 0, \cdots).$$

The smallest such ρ is denoted $\pi_\lambda(T)$. It can be shown that when F is a Banach space the space $\pi_\lambda(E, F)$ of all the absolutely λ-summing maps on E into F is a Banach space with the norm defined by $\pi_\lambda(\cdot)$.

The space λ is said to have the norm iteration property if for each sequence (x^n) of elements of λ we have $p[p(x^n)]=p[p(x_i)]$ where $x_i=(x_i^1, x_i^2, \cdots, x_i^n, \cdots)$. It is easily verified that the spaces c_0 and l^p $(1\leqq p\leqq\infty)$ have the above property.

THEOREM 9. *If λ has the norm iteration property then $N_\lambda(E, F)\subset\pi_\lambda(E, F)$ and $\pi_\lambda(T)\leqq N_\lambda(T)$.*

We remark now that Theorem 9 above is true also for quasi-λ-nuclear maps with practically the same proof as that of Theorem 9. In case $\lambda=l^p$ $(p\geqq 1)$ the results of Persson and Pietsch [5] show that by taking the composition product of a certain finite number of p-quasi-nuclear maps one can obtain ultimately a nuclear map. In a rather general set up as ours we cannot prove a result of that type. Consequently when one attempts to formulate the concept of a λ-nuclear space using the standard canonical mappings, one obtains naturally two related concepts, those of λ-nuclear spaces and of quasi-λ-nuclear spaces.

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