ON THE COHOMOLOGY OF STABLE TWO STAGE POSTNIKOV SYSTEMS

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Introduction. Let $\xi=(E, p, B, F)$ denote a two stage Postnikov system with stable k-invariant. We announce results about $H^*(\Omega E)$ as a Hopf algebra over the Steenrod algebra. Mod 2 cohomology is used exclusively. Unexplained notation is from [4] and [5]. I am grateful to D. Anderson, W. Massey, F. Peterson and H. Salomonsen for many useful remarks.

We make the following assumptions on ξ, in addition to those of [5, p. 38]. F and B are simply connected products of finitely many Eilenberg-MacLane spaces. The nonzero homotopy groups of the factors of B are infinite cyclic or cyclic of order 2^k, $k=1, 2, \cdots$. All factors of F have Z_2 (cyclic group of order 2) as their only nonzero homotopy group.

Results of [3], [4], [5] and [8] give $H^*(\Omega E)\cong R\otimes U(X')$. The isomorphism is as algebras over Z_2 and $\otimes$ is over Z_2. $R=R(\Omega\xi)=H^*(\Omega B)/\ker \Omega p^*$ and $X'=X'(\Omega\xi)$ is considered as known, [5, p. 54]. In general $H^*(\Omega E)$ does not split this way as a Hopf algebra over Z_2. The new result is Theorem A. It gives $H^*(\Omega E)$ as a coalgebra over R. It also gives information on the extension problem represented by the fundamental sequence of $\Omega\xi$ [5, p. 54]. This use of the Hopf algebra structure is well known, [1], [5] and [6].

1. **The main theorem.** Consider the following diagram of unstable A-modules and A-maps. The squares are commutative.

$$X'(\Omega\xi) \xleftarrow{\alpha} Y''/\lambda Y'' \xleftarrow{\pi} Y'' \xrightarrow{c} \Omega Y, \tag{1}$$

$$\begin{array}{ccccc} Y & \xrightarrow{f^*} & Z & \xrightarrow{\rho} & Z' \\ \sigma_{B_0}\downarrow & & \sigma_B\downarrow & & \sigma'\downarrow \\ \Omega Y & \xrightarrow[\Omega f^*]{} & \Omega Z & \xrightarrow[\rho']{} & \Omega Z' \end{array} . \tag{2}$$

Here α is an A-isomorphism of degree -1; π, ρ and ρ' are natural projections; c is inclusion, and σ' is the obvious map. The remaining

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maps and modules are as in [5, p. 63]. In particular $Y''=\ker \Omega f^*$, Z' and $\Omega Z'$ are coker f^* and coker Ωf^* respectively.

Using (1) and (2) we associate with each homogeneous element $x\in X'(\Omega\xi)$ an element $w\in\Omega Z'$ as follows. Let $y\in\Omega Y$ such that $\alpha^\pi(y)=x$. Let $t\in Y$ such that $\sigma_{B_0}(t)=y$. Since $\sigma_B f^*(t)=0$, $f^*(t)=\lambda z$ for some $z\in Z$. Let $w=\sigma'\rho(z)$. Note that the calculation of w just involves the Adem relations.

PROPOSITION 1. *w is a unique element of $\Omega Z'$.*

PROOF. σ_{B_0} is a map of degree -1 and λ doubles degrees. Hence $\rho f^*|\sigma_{B_0}^{-1}(\lambda Y'')=0$. This and looking at the choices involved in the definition of w give the result.

THEOREM A. *There exists an element $e\in P(\Omega\xi)$ such that $\Omega i^*(e)=x$ and $\bar\mu_2(e)=q(w\otimes w)$. Here q is the map*

$$\Omega p^*\otimes\Omega p^*: R(\Omega\xi\times\Omega\xi)\to P(\Omega\xi\times\Omega\xi).$$

The notation is [5, p. 63]. The proof uses the Serre spectral sequence in a manner similar to but more involved than arguments of [2] and [7].

REMARKS. 1. Theorem A amounts to calculating the homomorphism

$$X'(\Omega\xi)/\text{im }\sigma_3\to R(\Omega\xi\times\Omega\xi)/\text{im }\bar\mu_1$$

in the exact sequence at the bottom of p. 63 [5]. (μ_i should be replaced by $\bar\mu_i$, $i=1, 2$, there.)

2. If degree x is odd, then $w=0$. If degree x is even, it is quite possible for $w=0$ and not have $x\in\text{im }\sigma_3$. An example is given by $B=K(Z_2, 2)$, $F=K(Z_2, 7)$ and k-invariant $\text{Sq}^4\,\text{Sq}^2$. This example was also discovered by Massey.

3. Let $\{x_i\}$ be a homogeneous Z_2-basis for $X'(\Omega\xi)$. Let $\{e_i\}\subset P(\Omega\xi)$ satisfy Theorem A with $\Omega i^*(e_i)=x_i$. Then, by results of [4] and [5], $\{1\}\cup\{e_i\}$ form a simple system of generators for $H^*(\Omega E)$ as an algebra over R. Thus Theorem A calculates the coproduct of $H^*(\Omega E)$ considered as coalgebra over R. (R acts on $H^*(\Omega E)\otimes H^*(\Omega E)$ via $q\mu_1$.)

4. Let $\{x_i\}$ and $\{e_i\}$ be as in Remark 3. Let $\theta\in A$ and consider $\sum x_j=\Omega i^*(\theta e_i)$. Then $(\theta e_i+\sum e_j)=\Omega p^*(r)$ for a unique $r\in R$. The naturality of fundamental sequences with respect to loop multiplication and suspension gives much information about r. For example a unique element $[r]\in R/S$ is determined by the formula

$$q\bar\mu_1([r])=\bar\mu_2(\theta e_i+\sum e_j).$$

Here $S \subset R$ is the A-submodule of primitives and $\bar{\mu}_1 : R/S \rightarrow R \otimes R$ is considered as an A-map. It is well known to be a monomorphism. A similar formula can be obtained using suspension. We remark that if F and B are 2-connected and R is an exterior algebra over Z_2, then such formulae and the knowledge of $P(\Omega^2\xi)$ as an A-module permit a complete calculation of $P(\Omega\xi)$ as an A-module. We defer details to a longer paper.

References

1. J. F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. (2) **72** (1960), 20–104. MR **25** #4530.

2. J. R. Harper, *Stable secondary cohomology operations*, Comment. Math. Helv. **44** (1969), 341–353.

3. L. Kristensen, *On the cohomology of two-stage Postnikov systems*, Acta Math. **107** (1962), 73–123. MR **25** #4525.

4. W. S. Massey and F. P. Peterson, *The cohomology structure of certain fibre spaces*. I, Topology **4** (1965), 47–65. MR **32** #6459.

5. ———, *The mod 2 cohomology structure of certain fibre spaces*, Mem. Amer. Math. Soc. No. 74 (1967). MR **37** #2226.

6. R. J. Milgram, *The cohomology over the Steenrod algebra of some two stage Postnikov systems*, Quart. J. Math. **20** (1969), 161–169.

7. F. P. Peterson, *A note on H-spaces*, Bol. Soc. Mat. Mexicana (2) **4** (1959), 30–31. MR **22** #11393.

8. L. Smith, *The cohomology of stable two stage Postnikov systems*, Illinois J. Math. **11** (1967), 310–329. MR **34** #8406.

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