PRODUCT FORMULAS FOR $L_n(\pi)$

BY JULIUS L. SHANESON

Communicated by William Browder, December 12, 1969

Introduction. In this note we prove some product formulas for non-simply-connected even dimensional surgery obstructions. This complements [8] (and in fact uses [8] as well as [5]). We also give a simple example of the type of geometric construction that product formulas make possible.

1. **Product formulas.** Let Ω_m be the oriented cobordism classes of oriented, closed, smooth or piecewise-linear (P.L.) manifolds of dimension m. Let π be a finitely presented group, let $w:\pi\to Z_2$ be a homomorphism, and let $L_n^h(\pi, w)$ be the Wall surgery obstruction group for the homotopy problem in dimension $n\geqq 5$ (see [6] or [7]). That is, if $(X^n, \partial X)$ is a manifold, if ξ is a vector bundle over X, if $f:(M, \partial M)\to(X, \partial X)$ is a map of degree one whose restriction induces a homotopy equivalence of boundaries, and if F is a stable framing of $\tau(M)\oplus f^*\xi$; then if $(\pi_1 X, w^1 X)=(\pi, w)$, there is an obstruction $\theta(M, f, F)$ in $L_n^h(\pi, w)$ that vanishes if and only if (M, f, F) is cobordant relative the boundary to (N, g, G), g a homotopy equivalence. The Wall groups satisfy $L_n^h(\pi, w)=L_{n+4}^h(\pi, w)$, and surgery obstructions are invariant under products with complex projective space CP^2. For $n\geqq 6$, every element can be realized as $\theta(M, f, F)$ for a suitable given X and ξ; e.g. $X=K\times I$ and $\xi=\nu(X)$, the normal bundle of X. For low dimensions, obstructions are defined by crossing with CP^2; their vanishing is a necessary condition for the surgery problem to be solvable.

There is a pairing

$$\Omega_m \times L_n^h(\pi, w) \to L_{n+m}^h(\pi, w)$$

defined as follows: Let $\alpha\in\Omega_m$ and let $z\in L_n^h(\pi, w)$. Assume $n\geqq 6$. Choose a simply-connected manifold P representing α, and let X, ξ, M, f, and F be as above so that $\theta(M, f, F)=z$. Let G be the natural framing of $\tau(P)\oplus\nu(P)$, $\nu(P)$ a high dimensional normal bundle of P. Then we make the definition

$$\alpha \times z = \theta(P\times M, 1\times f, G\times F).$$

AMS Subject Classifications. Primary 5705, 5710; Secondary 5747.

Key Words and Phrases. Surgery, surgery obstruction, index, Whitehead torsion, nonlinear representation.

This is a well defined bilinear pairing; see §9 of [7].

A similar pairing exists for the obstruction groups $L_n^s(\pi, w)$ for the simple homotopy problem.

Let $I:\Omega_m \to \boldsymbol{Z}$ be the index homomorphism; i.e. $I(\alpha)=0$ if $m \not\equiv 0$ (mod 4) and $I(\alpha)=\text{Index } P$ for any $P\in\alpha$ if $m=0$ (mod 4), in which case the index of P is the index of the quadratic form $(x\cup y)[P]$ on $H^{m/2}(P; Q)$. Williamson [8] has shown that for n odd and m even,

$$\alpha \times z = I(\alpha)z, \qquad \alpha \in \Omega_m, \quad z \in L_n^s(\pi, w).$$

THEOREM 1.1. *Let m and n be even. Let $\alpha\in\Omega_m$, $z\in L_n^h(\pi, w)$. Then $\alpha\times z=I(\alpha)z$.*

PROOF. Let P be a simply-connected representative of α. Assume $n\geqq 6$. Let $\theta(M, f, F)=z$, where $f:(M, \partial_- M, \partial_+ M) \to (K\times I, K\times 0, K\times 1)$ with $f|\partial_- M:\partial_- M\to K\times 0$ a diffeomorphism (or P.L. equivalence), and with F a stable framing of $\tau(M) \oplus f^*\nu(K\times I)$. Let

$$j(K): L_n^h(\pi, w) \to L_{n+1}^s(\pi \times \boldsymbol{Z}, w_1)$$

be the map defined in §5 of [5]. It is clear from the definition that

$$(*) \qquad P \times j(K)z = j(P\times K)(\alpha \times z).$$

The formula of Williamson, applied to the left side of (*), gives zero if $m\equiv 2$ (mod 4). Hence $\alpha\times z=0$ also, since $j(P\times K)$ is monic by Theorem 5.1 of [5].

Suppose $m\equiv 0$ (mod 4). It follows from §9 of [8] that $j(K)$ depends only upon how we identify $\pi_1 K$ with π; i.e. only upon the choice of a map $K\to K(\pi, 1)$ that induces an isomorphism of fundamental groups. (This observation clears up a question raised in some remarks in §5 of [5]. In the present situation we could avoid this observation by making an extra geometric construction.) Hence, the formula of Williamson implies that the left side of (*) is $j(P\times K)I(\alpha)z = j(\mathrm{CP}^2\times K)I(\alpha)z$. So by 5.1 of [5] again, $\alpha\times z=I(\alpha)z$, which completes the proof.

Let $A_j(\pi, w)$, $j\geqq 0$, be the subquotient of the Whitehead group defined in §4 of [5].

COROLLARY 1.2. *Suppose $A_{n+1}(\pi, w)=0$, n even. Let $\alpha\in\Omega_m$, $z \in L_n^s(\pi, w)$, m even. Then $\alpha\times z=I(\alpha)z$.*

PROOF. By Proposition 4.1 of [5], the natural map of $L_n^s(\pi, w)$ to $L_n^h(\pi, w)$ is a monomorphism.

COROLLARY 1.3. *For m and n even, $\alpha\in\Omega_m$, $z\in L_n^s(\pi, w)$, $\alpha\times z-I(\alpha)z$ always has order two.*

PROOF. Every element of $A_{n+1}(\pi, w)$ has order two, for any π. By 1.1 and 4.1 of [5], $\alpha\times z-I(\alpha)z$ is in the image of the natural map $A_{n+1}(\pi, w)\to L_n^s(\pi, w)$.

REMARKS. (1) For π any finite Abelian group and w trivial, $A_{n+1}(\pi, w)=0$ if n is even.

(2) For $\pi=Z_n$ and w trivial, one can prove 1.2 using the idea of [4] to study the Wall groups via the Atiyah-Singer index theorem.

(3) Using Proposition 4.6 of [5] and the product formulas for Whitehead torsion, it is not hard to see that for $m\equiv 0 \pmod 4$ we have a commutative diagram with exact rows:

$$\begin{array}{ccccc} A_{n+1}(\pi, w) & \to & L_n^s(\pi, w) & \to & L_n^h(\pi, w) \\ \downarrow \lambda & & \downarrow \beta & & \downarrow \xi \\ A_{n+1}(\pi, w) & \to & L_n^s(\pi, w) & \to & L_n^h(\pi, w) \end{array}$$

where $\xi(z)=I(\alpha)z$, $\lambda(z)=I(\alpha)z$, and $\beta(z)=\alpha\times z$. The rows are part of Rothenberg's sequence (Proposition 4.1 of [5]). Thus to show that the congruence of 1.3 is an exact equality, it remains to solve an extension problem. A similar remark holds for $m\equiv 2 \pmod 4$.

2. **An application to nonlinear representations.** Theorem 1.1 and its corollaries can be used to construct various exotic manifolds, group actions, etc. For example, see [1] for some applications of the (previously well-known) simply-connected case. In this section we give one simple example of how to create a nonlinear representation by killing a surgery obstruction using 1.1.

Let G and H be compact Lie groups and let ρ be a smooth action of $G\times H$ on a closed manifold M, with isotropy subgroups G, H, and $\{e\}$. Then the fixed point set of G, $F(G)$, is invariant under H, and so we get an action $\alpha(\rho)$ on $F(G)$ by H. Similarly we have an action $\beta(\rho)$ on $F(H)$ by G.

Let λ be a free action of G on a homotopy sphere Σ^{2k-1}. We say λ is *normally linear* if there is a free linear (orthogonal) action μ on S^{2k-1} and a homotopy equivalence $h: S^{2k-1}/\lambda\to S^{2k-1}/\mu$ with vanishing normal invariant in $[S^{2k-1}/\mu; G/O]$. Let G_0 be the component of the identity of G and let $\pi=G/G_0$. Suppose $\pi\neq\{e\}$ and k is even or the smallest prime dividing $|\pi|$ is not two. Suppose dim G is even and $(2k-\dim G)\geqq 6$. Then, if a free linear action μ exists, it follows from results of Petrie [4] that there are infinitely many normally linear actions on the *standard* sphere that are P.L. (and even topologically) distinct.

THEOREM 2.1. *Let G be a compact, even dimensional Lie group, and let λ be a free normally linear action of G on a homotopy sphere Σ^{2k-1}, $k \geqq 2$. Let H_1 be either the group of unit complex numbers or unit quaternions, and let δ be the (usual) free linear representation of H_1 on $S^{4\epsilon m-1}$, $m \geqq 1$, $\epsilon = 1$ or 2 depending on whether* dim $H_1 = 1$ *or* 3. *Assume* $4\epsilon m + 2k - \dim G - 3 \geqq 5$. *Let H be any closed subgroup of H_1. Then $\exists$ a fixed point free action ρ of $G \times H$ on a homotopy sphere $M^{2k+4\epsilon m-1}$ with isotropy subgroups G, H, and $\{0\}$, so that $\alpha(\rho) = \delta | H$ and $\beta(\rho) = \lambda$.*

PROOF. Let $f\colon \Sigma^{2k-1}/\lambda \to S^{2k-1}/\mu = K$, μ a free orthogonal action of G, be a homotopy equivalence with vanishing normal invariant. Then there is a cobordism W with $\partial_+ W = S^{2k-1}/\lambda$, a map $\phi\colon (W, \partial_- W, \partial_+ W) \to (K \times I, K \times 0, K \times 1)$ of degree 1 with $\phi | \partial_+ W = f$ and $\phi | \partial_- W$ a diffeomorphism, and a stable framing F of $\tau(W) \oplus \phi^* \nu(K \times I)$. Let $z = \theta(W, \phi, F) \in L_n(\pi)$, $\pi = G/G_0$, G_0 the component of the identity element of G, $n = 2k - 1 - \dim G$. (We omit w from the notation since it is trivial here.)

The quotient $Q = S^{4\epsilon m-1}/\delta$ is either the complex projective space CP^{2m-1} or the quaternionic projective space HP^{2m-1}. Both have index zero. Hence $[Q] \times z = 0$. It follows (using the periodicity of surgery obstructions for $n \leqq 4$) that there is an h-cobordism U of $K \times Q = \partial_- U$ to $(\Sigma^{2k-1}/\lambda) \times Q = \partial_+ U$ and a map $g\colon U \to K \times I \times Q$ so that the restriction $g | \partial_- U \colon \partial_- U \to K \times 0 \times Q$ is the identity and $g | \partial_+ U \colon \partial_+ U \to K \times 1 \times Q$ is $f \times 1$.

Now $K \times I \times Q$ is the base space of a principal $G \times H_1$-bundle with total space $S^{2k-1} \times S^{4\epsilon m-1} \times I$; the action is just $(\mu \times \delta) \times I$. Let V be the total space of the bundle induced over U via g from this bundle. Then V is an h-cobordism from $S^{2k-1} \times S^{4\epsilon m-1}$ to $\Sigma^{2k-1} \times S^{4\epsilon m-1}$ and carries a free $G \times H_1$ action, ξ.

Let

$$M = D^{2k} \times S^{4\epsilon m-1} \cup V \cup \Sigma^{2k-1} \times D^{4\epsilon m};$$

i.e. take the disjoint union and identify $\partial_- V$ with $\partial(D^{2k} \times S^{4\epsilon m-1})$ and $\partial_+ V$ with $\partial(\Sigma^{2k-1} \times D^{4\epsilon m})$. Since μ and δ are orthogonal they extend to actions $\bar{\mu}$ and $\bar{\delta}$ on D^{2k} and $D^{4\epsilon m}$, respectively, fixed point free except at the origin. The union $\rho = (\bar{\mu} \times \delta) \cup \xi \cup (\lambda \times \bar{\delta})$ is an action of $G \times H_1$ on M. It is easy to verify that M is a homotopy sphere and that $\rho | G \times H$ has the desired properties.

Note. For the special case of $Z_p \times Z_q$, we can make $\alpha(\rho)$ and $\beta(\rho)$ arbitrary normally linear actions of Z_q on S^{4m-1} and Z_p on S^{4k-1}, respectively. The argument is similar.

REFERENCES

1. W. Browder, *Surgery and the theory of differentiable transformation groups*, Proc. Conf. Transformation Groups, Tulane Univ., New Orleans, La., 1967.

2. ———, *Surgery on simply-connected manifolds* (to appear).

3. K. W. Kwun and R. H. Szczarba, *Product and sum theorems for Whitehead torsion*, Ann. of Math. (2) **82** (1968), 183–190. MR **32** #454.

4. T. Petrie, *The Atiyah-Singer invariant, the Wall groups $L_n(\pi, 1)$, and the function te^x+1/te^x-1*, Ann. of Math. (to appear).

5. J. L. Shaneson, *Wall's surgery obstruction groups for $Z \times G$*, Ann. of Math. (2) **90** (1969), 296–334.

6. C. T. C. Wall, *Surgery of non-simply-connected manifolds*, Ann. of Math. (2) **84** (1966), 217–276. MR **35** #3692.

7. ———, *Surgery of compact manifolds*, Academic Press (to appear).

8. R. E. Williamson, *Surgery in $M \times N$ with $\pi_1 M \neq 1$*, Bull. Amer. Math. Soc. **75** (1969), 582–585.

PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540