ON A CONJECTURE IN THE THEORY OF PERMANENTS[1]

BY ALBERT NIJENHUIS AND HERBERT S. WILF

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Let A be an $n \times n$ matrix of zeros and ones, with r_i ones in the ith row $(i=1, \cdots, n)$. It has been conjectured [1] that

$$\text{(1)} \qquad \operatorname{Per} A \leqq \prod_{i=1}^{n} r_i!^{1/r_i}$$

which, if true, would be best possible. We sketch here a proof of the fact that

$$\text{(2)} \qquad \operatorname{Per} A \leqq \prod_{i=1}^{n} \{r_i!^{1/r_i} + \tau\}$$

where $\tau = .1367 \cdots$ is a universal constant. Details of the proof will appear elsewhere [2].

Suppose ϕ is a function of the positive integers for which $\phi(1)=1$ and

$$\text{(3)} \qquad \operatorname{Per} A \leqq \prod_{i=1}^{n-1} \phi(r_i)$$

is true for all $(n-1)\times(n-1)$ matrices A. If now A is $n \times n$, expanding by minors down some column, one finds that (3) holds with n replacing $n-1$ provided that

$$\text{(4)} \qquad \sum_{i=1}^{c} \frac{1}{\phi(r_i - 1)} \prod_{k=1}^{c} \frac{\phi(r_k - 1)}{\phi(r_k)} \leqq 1$$

for all positive integers c and $r_1, \cdots, r_c \geqq 2$. Consider the function ϕ defined recursively by

$$\text{(5)} \qquad \begin{aligned} &\text{(a)} \quad \phi(1) = 1 \\ &\text{(b)} \quad \phi(n+1) = \phi(n) \exp[1/e\phi(n)] \qquad (n \geqq 1). \end{aligned}$$

Substituting (5) into (4) one finds easily that (4) holds.

By rather lengthy arguments we prove that for the ϕ of (5) we have

$$\text{(6)} \qquad \phi(n) = \frac{n}{e} + \frac{\log n}{2e} + \frac{A}{e} + o(1) \qquad (n \to \infty)$$

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and

$$\phi(n) \leqq n!^{1/n} + \frac{A - \log \sqrt{(2\pi)}}{e} \qquad (n = 1, 2, 3, \cdots) \tag{7}$$

which together prove (2) with $\tau = (A - \log \sqrt{(2\pi)})/e$.

References

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UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19104