A NONSOLVABLE GROUP OF EXPONENT 5

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THEOREM 1. *There exists a group $\mathcal{G}$ of exponent 5 which is locally nilpotent, but not nilpotent. In particular, $\mathcal{G}$ is not solvable.*

Thus there exist varieties which are nonsolvable, but locally finite and locally solvable.

To prove Theorem 1, we first show that a certain ring is not nilpotent. Let R be the free associative ring of characteristic 5 generated by noncommuting indeterminates $x_1, x_2, \cdots$, and let L be the Lie ring in R generated by $x_1, x_2, \cdots$ where addition in L is the same as in R and Lie multiplication is commutation $[x, y]=xy-yx$ in R. An element of L will be called a Lie element.

THEOREM 2. *If we impose on R the following identical relations for Lie elements x and y:*

$$x^3 = 0 \tag{i}$$

and

$$x^2y - 3xyx + 3yx^2 = 0 \tag{ii}$$

then the resulting ring is not nilpotent.

REMARK. Higgins in [3] showed that (i) and (ii) holds in the endomorphism ring of the additive group of a Lie ring satisfying the third Engel condition.

Also worth mentioning is the following result which is equivalent to Theorem 2 as shown in Walkup [8].

THEOREM 3. *There exists a Lie ring of characteristic 5 which satisfies the third Engel condition and which is not nilpotent.*

G. Higman [4] and A. I. Kostrikin [5] showed that a Lie ring of characteristic 5 satisfying the fourth Engel condition is locally nilpotent, and in view of Theorem 3, this is the best one can say.

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Kostrikin [6], in fact, was able to prove the very general theorem that a Lie ring satisfying the nth Engel condition and having prime characteristic $p>n$ is locally nilpotent.

In getting away from the finite generation condition, P. Higgins [3] and Heineken [2], showed that an associative ring with characteristic prime to 2, 3, 5 and 7 in which the cube of every Lie element is zero must be nilpotent of index at most 3^9. D. Walkup [8] in his thesis improved this result in two ways. First he showed that no restriction on the prime 7 is necessary and secondly that the nilpotency index can be greatly reduced. Specifically he showed

THEOREM 4. *Let R' be the free associative ring generated by noncommuting indeterminates $x_1, x_2, \cdots$, with coefficients in a ring in which division by* 2, 3, *and* 5 *are possible and the cubes of all Lie elements are zero. Then R' is nilpotent of index at most* 9.

We sketch the proof of Theorem 2 and the deduction of Theorem 1 from it.

Relations (i) and (ii) together are equivalent to the "Higgins relations",

$$\text{(iii)} \qquad xyz + xzy = yzx + zyx = 2(yxz + zxy)$$

for all (homogeneous) Lie elements x, y, z.

Let H be the ideal of R generated by (iii). Denote by R_n the vector subspace of R with basis consisting of all monomials of total degree $2n$ and degree 2 in each indeterminate x_i, $i=1, 2, \cdots, n$. Making use of the relations (iii), we are able to establish inductively for each $n \geqq 2$ the existence of a linear transformation α of R_n onto Z_5, the integers modulo five, which satisfies:

(a) $\alpha(x_1^2 x_2^2 \cdots x_n^2)=1$.

(b) $\alpha(MN)=\alpha(NM)$, where M and N are any monomials such that MN is in R_n.

(c) $\alpha(M)=\alpha(M')$, where M is any monomial in R_n and M' is the monomial obtained from it by permuting (the names of) the indeterminates.

(d) $\alpha(M)=\alpha(M^T)$, where M is any monomial in R_n and M^T is the monomial obtained from it by reversing the order of the factors (of degree 1).

(e) $\alpha[M(xyz+xzy)]=\alpha[M(yzx+zyx)]=2\alpha[M(yxz+zxy)]$, where x, y, and z are chosen from among the generators x_i and M is any monomial such that the indicated products are in R_n.

We then show that the kernel of α, say S_n, contains $H \cap R_n$, i.e., (e) holds for all Lie elements x, y and z such that $Mxyz$ is in R_n.

Since $x_1^2x_2^2 \cdots x_n^2$ is not in S_n and hence not in H, R is *not* nilpotent modulo H, establishing Theorem 2.

Using Bruck's notation in §3 of [1], we can show that Theorem 2 implies that R is not nilpotent modulo the permutation ideal of T_k. Then, by Theorem 4.3 of [1] this last fact implies the negation of the statement $R(3, \pi)$ in Bruck's notes [1], i.e.,

THEOREM 5. *There exists a group ring Z_5G over the field Z_5 of integers modulo 5 such that the augmentation ideal of Z_5G is not nilpotent modulo the ideal I generated by all elements $(g-1)^3$ with g in G.*

To complete the proof of Theorem 1, we use a standard construction. Let Z_5G and I be as in Theorem 5. Define group $\mathcal{G}$ to be the set of all ordered pairs $\{g, r\}$, $g \in G$, $r \in Z_5G/I$ with the multiplication

$$\{g, r\}\{h, s\} = \{gh, rh + s\}.$$

An easy check shows that $\mathcal{G}$ has exponent 5. If $a = \{1, 1\}$ and $b_i = \{g_i, 0\}$, then the commutator

$$(a, b_1, b_2, \cdots, b_n) = \{1, (g_1 - 1)(g_2 - 1) \cdots (g_n - 1)\}.$$

Since the augmentation ideal of Z_5G is not nilpotent modulo I, $\mathcal{G}$ is not nilpotent, and hence by a theorem of Tobin [7], $\mathcal{G}$ is not solvable. Thus, G is also nonsolvable.

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