ON PERIODIC SOLUTIONS OF NONLINEAR HYPERBOLIC EQUATIONS AND THE CALCULUS OF VARIATIONS

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Let G be a bounded domain in R^N with boundary ∂G. Then the system (for $p(x)$ a strictly positive $C'(\overline{G})$ function)

$$
\begin{aligned}
p(x)\, u_{tt} - \Delta u &= 0 \qquad (\text{in } G), \\
u/\partial G &= 0,
\end{aligned}
\tag{1}
$$

has a countably infinite number of distinct periodic solutions (i.e. "normal modes"). In this note we shall show that the same conclusion can be established for the nonlinear system

$$
\begin{aligned}
p(x)\, u_{tt} - \Delta u + f(x, u) &= 0, \\
u/\partial G &= 0,
\end{aligned}
\tag{2}
$$

under certain restrictions on the functions $f(x, u)$ and $p(x)$. (Throughout we assume $f(x, 0) \equiv 0$, so that $u(x, t) \equiv 0$ satisfies (2).) Furthermore similar results can be obtained for higher order systems in which the Laplace operator Δ is replaced by a strongly elliptic operator of order $2m$ and the boundary conditions are suitably altered (such systems occur in the theory of elastic vibrations).

Our proofs are based on approximating the system (2) by a Hamiltonian system of ordinary differential equations, as in [**4**]. The periodic solutions of the associated Hamiltonian systems are then investigated by the methods of the calculus of variations in the large, as studied by the author in [**1**]. Periodic solutions of the original system (2) are then obtained by taking limits. Previous mathematical studies of periodic solutions of (2) (e.g. [**2**], [**3**], [**5**]) have been primarily perturbation results and have not considered the totality of periodic solutions of (2).

1. **Preliminaries.** Let x denote a point in G and $W_{1,2}(G_T)$ denote the Sobolev space of functions $u(x, t)$, T-periodic in t, which are square integrable and possess square integrable derivatives over $G \times [0, T]$. By $\dot{W}_{1,2}(G_T)$ we denote the subspace of $W_{1,2}(G_T)$ consisting of functions which vanish on ∂G (in the generalized sense). $\dot{W}_{1,2}(G_T)$ is a Hilbert space with respect to the inner product

$$(u, v)_{1,2}^{T} = \int_0^T \int_G \{u_t v_t + \operatorname{grad} u \cdot \operatorname{grad} v\}.$$

By a T-periodic weak solution of (2), we understand a function $u(x, t) \in \dot{W}_{1,2}(G_T)$ which satisfies the following integral identity for all $\phi \in \dot{W}_{1,2}(G_T)$:

$$0 = \int_0^T \int_G \{u_t \phi_t - \operatorname{grad} u \cdot \operatorname{grad} \phi - f(x, u)\phi\}. \tag{3}$$

Henceforth we shall study T-periodic weak solutions $u(x, t) \not\equiv 0$.

2. **Statement of results.** First we discuss the existence of a global one-parameter family of periodic solutions of (2) whose frequency corresponds roughly to the lowest eigenvalue λ_1^2 of Δ on G. To this end, we assume that (i) $f(x, u)$ satisfies the following growth condition:

$$(*) \qquad |f(x, u)| \leqq K_0 |u|, \text{ for } |u| \text{ sufficiently large}$$

and $F(x, u) \leqq K_1 f(x, u)u$ where $F(x, u) = \int_0^u f(x, t)dt$ and K_0, K_1 are positive constants independent of u and x (ii) $(\mathrm{I}_i)\lambda_j/\lambda_i =$ integer is satisfied for at most finitely many indices j, where λ_j^2 denote the eigenvalues of Δ on G with respect to $p(x)$.

THEOREM 1. *Let $f(x, t)$ be a locally Lipschitz continuous function, odd in t, satisfying the conditions* $(*)$ *and* (I_1) *and such that for all real t and $x \in G$, $tf(x, t) \geqq 0$. Then the system* (2) *has a one-parameter family of distinct $T_1(R)$-periodic weak solutions $\tilde{u}_1(R)$ where the parameter R is sufficiently small, provided $f(x, t) = o(t)$ for small t. In addition, R and $T_1(R)$ are related by*

$$2\pi R = T_1(R) \int_0^{T_1(R)} \int_G \left(\frac{\partial \tilde{u}_1(R)}{\partial t}\right)^2. \tag{4}$$

Furthermore as $R \to 0$, $T_1(R) \to 2\pi\lambda_1^{-1}$, where λ_1^2 is the smallest eigenvalue of Δ.

The next result concerns the existence of an infinite number of distinct one-parameter families of periodic solutions of (2), whose frequencies correspond roughly to the other eigenvalues of Δ on G.

THEOREM 2. *Let $f(x, t)$ be a locally Lipschitz function of x and t, odd in t, satisfying the conditions* $(*)$ *and* (I_i), *and such that $f(x, t) = o(t)$ for small t. Then the system* (2) *has, for sufficiently small R, an infinite* *number of distinct one-parameter families of $T_i(R)$-periodic weak solutions $u_i(R)$ $(i=1, 2, \cdots)$, where R is defined by* (4). *Furthermore as $R\to 0$, $T_i(R)\to 2\pi\lambda_i^{-1}$ where λ_i^2 denote the eigenvalues of Δ with respect to $p(x)$ ordered by magnitude.*

3. **Outline of proofs.** Denote the eigenvalues (ordered by magnitude) and eigenfunctions of Δ with respect to $p(x)$ by λ_i^2 and $u_i(x)$, respectively. Then approximate weak solutions of (2) by functions of the form $\tilde{u}_n = \sum_{i=1}^n q_i^{(n)}(t)u_i(x)$, where $q_i^{(n)}(t)$ are functions to be determined. Substituting $\tilde{u}_n$ in (2) we find the following approximate equations for $q_i^{(n)}(t)$.

$$\text{(5)}\quad \ddot{q}_i^{(n)} + \lambda_i^2 q_i^{(n)} + \int_G pf\left(x, \sum_{j=1}^n q_j^{(n)} u_j\right) u_i = 0 \qquad (i = 1, \cdots, n).$$

Setting $t=\sigma s$, we consider 2π-periodic solutions of the system

$$\text{(6)}\quad \ddot{q}_i^{(n)} + \sigma^2\left[\lambda_i^2 q_i^{(n)} + \int_G pf\left(x, \sum_{j=1}^n q_j^{(n)} u_j\right) u_i\right] = 0 \;\; (i = 1, \cdots, n).$$

The 2π periodic solutions of (6) can be regarded as critical points of the functional

$$\mathcal{G}_n(q^{(n)}) = \int_0^{2\pi}\left\{\sum_{i=1}^n \lambda_i^2 (q_i^{(n)})^2 + \int_G p\,2F\left(x, \sum_{i=1}^n q_i^{(n)} u_i\right)\right\}$$

over the admissible class of odd, 2π periodic n-vector functions $q^{(n)} = (q_1^{(n)}, q_2^{(n)}, \cdots, q_n^{(n)})$ such that $\int_0^{2\pi}(\dot{q}^{(n)})^2 = R$, a positive constant. We compare these critical points with the critical points of the "linearized" problem $V_R(n)$: i.e. the critical points of the functional

$$Q_n(q^{(n)}) = \int_0^{2\pi} \sum_{i=1}^n \lambda_i^2 (q_i^{(n)})^2$$

over the same admissible class of functions as above. The critical values of $V_R(n)$ are proportional to λ_i^2/K^2, $(i=1, 2, \cdots, n)$, K an integer. In the following we order these critical values in decreasing order of magnitude and denote the critical value proportional to λ_i^2 as the $n(i)$th number in this ordering (with multiplicities included).

The proofs make use of

(i) the techniques of the Ljusternik-Schnirelmann theory of critical points on Hilbert manifolds for fixed n, and

(ii) a limiting selection procedure as $n\to\infty$.

Consider the set of continuous odd 2π periodic n-vector functions

$q^{(n)}(s)$ which possess a square integrable generalized derivative. This set forms a Hilbert space H^n with norm

$$\|q^{(n)}\|^2 = \sum_{i=1}^{n} \int_0^{2\pi} \dot{q}_i^2(s)\,ds.$$

In H^n we identify the antipodal points on the sphere $S^n(R) = \{q^{(n)} \mid \|q^{(n)}\|^2 = R\}$ to obtain the infinite dimensional real projective space $P_R^\infty(H^n)$, and by virtue of the evenness of $\mathcal{G}_n(q^{(n)})$, we consider $\mathcal{G}_n(q^{(n)})$ as defined on $P_R^\infty(H^n)$. To prove Theorem 1, we consider the variational problem

$$V(n, 1)\colon\ c_{n(1)}(R) = \sup_{[W]_{n(1)}} \min_{W} \mathcal{G}_n(q^{(n)})$$

where

$$[W]_{n(1)} = \{W \mid W \subset P_R^\infty(H^n),\ \mathrm{cat}(W, P_R^\infty(H^n)) \geqq n(1)\}.$$

The problem $V(n, 1)$ has a solution $q_i^{(n)}(s)$ which is a critical point of $\mathcal{G}_n(q^{(n)})$ and so satisfies (6) for some $\sigma_{n(1)}(R)$, thus giving rise to a $2\pi\sigma_{n(1)}(R)$ periodic solution of (5). As R runs through small positive values, one-parameter families of periodic solutions of (5) are generated; as in [1]. Now as $n \to \infty$ for fixed R, the sequences $\{c_{n(1)}(R)\}$, $\{\sigma_{n(1)}(R)\}$ and $\{\| \sum_{i=1}^n u_i(x) q_i^{(n)}(s)\|_{1,2}^{2\pi}\}$ are uniformly bounded, so that (after suitable reindexing) there are subsequences $\tilde{u}_n(x, s) = \sum_{i=1}^n u_i(x) q_i^{(n)}(s)$ and $\sigma_{n(1)}(R)$ such that $\tilde{u}_n(x, s) \to \tilde{u}(x, s)$ weakly in $\dot{W}_{1,2}(G_{2\pi})$ and $\sigma_{n(1)}(R) \to \sigma_1(R)$ where $\int_0^{2\pi}\int_G (\partial\tilde{u}/\partial s)^2 = R$. Hence $\tilde{u}(x, \sigma_1^{-1}(R)t)$ is a $2\pi\sigma_1(R)$ periodic weak solution of (2), and as $R \to 0$, one finds $\sigma_1(R) \to 1/\lambda_1$ provided $f(x, u) = o(u)$. Theorem 2 follows by replacing $n(1)$ by $n(i)$ $\{i = 2, 3, 4, \cdots\}$ in the above argument. The existence of the above limits is proven by using conditions (*) and (I.) to find a priori bounds on $\tilde{u}_n$ and its derivatives that are independent of n.

4. Extensions. (1) Let A be a strongly elliptic linear selfadjoint operator of order $2m$. Then the methods used to prove Theorems 1 and 2 can be applied to study the periodic solutions of the system

$$u_{tt} + (-1)^m A u + f(x, u, Du, \cdots, D^{2m-2}u) = 0 \qquad \text{in } G$$

$$D^{\alpha} u \,|_{\partial G} = 0 \qquad |\alpha| \leqq m - 1$$

provided $f(x, u, \cdots, D^{2m-2}u)$ satisfies suitable positivity and growth conditions and is derivable as an Euler-Lagrange expression of a functional $\int_G F(x, u, \cdots, D^{m-1}u)$.

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