AXIOM A+NO CYCLES$\Rightarrow\zeta_f(t)$ RATIONAL

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Throughout, $f: M\to M$ is a smooth diffeomorphism of a compact C^∞ manifold without boundary.

Let N_i denote the number of fixed points of f^i. Then

DEFINITION. $\zeta_f(t)=\exp\sum_{i=1}^\infty N_i t^i/i$ (as a formal power series in t). This definition, due to Artin-Mazur [1], is inspired by Weil's zeta function for a variety defined over a finite field [6]. For the connection of Weil's zeta function with the Riemann zeta function, see [3].

Recall the following definitions from *Differentiable dynamical systems* [4].

DEFINITION. $x\in M$ is *nonwandering* if for every neighborhood U of x, there is an $n>0$ such that $f^n(U)\cap U\neq\emptyset$. $\Omega(f)=\Omega$ is the set of nonwandering points of f. Ω is closed.

DEFINITION. f satisfies Axiom A if $T_\Omega(M)$ has a continuous splitting $T_\Omega(M)=E^s+E^u$, invariant under Tf, such that there exist positive constants c, λ, $\lambda<1$ satisfying the inequalities

$$\|Tf^n v\| \leqq c\lambda^n\|v\| \quad \text{if } n>0 \text{ and } v\in E^s,$$

$$\|Tf^n v\| \geqq c\lambda^{-n}\|v\| \quad \text{if } n>0 \text{ and } v\in E^u.$$

Furthermore, it is assumed that the periodic points of f are dense in Ω.

If f satisfies Axiom A, then $\Omega=\Omega_1\cup\cdots\cup\Omega_k$ where Ω_i is invariant under f and $f|\Omega_i$ is topologically transitive. Define the relation $\geqq$ by $\Omega_i\geqq\Omega_j$ if $W^u(\Omega_i)\cap W^s(\Omega_j)\neq\emptyset$. Here $W^u(\Omega_i)$ is the set of points tending toward Ω_i under negative iteration; $W^s(\Omega;\ )$ is the set of points tending toward Ω_j under iteration.

DEFINITION. If f satisfies Axiom A and the relation $\geqq$ defined above is a partial ordering, then f is said to have the *No Cycle Property*.

The purpose of this paper is to prove the following:

THEOREM. *If f satisfies Axiom* A *and the No Cycle Property, then $\zeta_f(t)$ is rational.*

The basic idea of the proof is due to Williams [7]. As a preliminary,

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recall the following well-known lemma about formal power series.

LEMMA. *$\exp(\sum N_i t^i/i)$ is a rational function of t if and only if there exist integral matrices A and B such that $N_i = \mathrm{Tr} A^i - \mathrm{Tr} B^i$.*

PROOF. See Williams [8].

Also recall the Lefschetz trace formula [2]: if X is a Euclidean neighborhood retract with boundary and $g: X \to X$ is a continuous map with image in the interior of X, then

$$\sum_{p \in \mathrm{fix}\, g} L(p) = \sum_i (-1)^i \mathrm{Tr}\, g_{*i}.$$

Here fix g is the set of fixed points of g, $L(p)$ is the index of g at p, g_{*i} is the induced map of $H_i(X; Q)$.

PROPOSITION (SMALE [4]). *If $f: M \to M$ satisfies Axiom A and p is a fixed point of f then $L(p) = (-1)^u \Delta$ where $u = \dim E^u_p$ and $\Delta = +1$ or -1 as $Tf|_{E^u_p}$ preserves or reverses orientation respectively.*

The reader is advised to compare our proof of the theorem with Williams' proof that the zeta function of an attractor is rational [7].

PROOF OF THEOREM. Let Ω_0 be one of the closed invariant subsets of Ω on which f is topologically transitive. We show that $f|\Omega_0$ has a rational zeta function. Lemma 4.1 of [5] shows that we can find an arbitrarily small neighbourhood U of Ω_0 and an open set U_0 such that

(1) U and U_0 have smooth boundaries,
(2) $f(\overline{U} \cup \overline{U}) \subset U \cup U_0$,
(3) $f(U_0) \subset U_0$,
(4) $\overline{U}_0 \cap \Omega_0 = \varnothing$.

In particular we choose U, U_0 as above so that the bundle $E^u|\Omega_i$ has a continuous extension to $\overline{U}$.

Set $V = \overline{U} - (U_0 \cap U)$. V is a compact neighborhood of Ω_0 and there is a continuous extension of $E^u|\Omega_0$ to V, which we again denote E^u. There is a double cover $\pi: \tilde{V} \to V$ such that the bundle $\tilde{E}^u$ induced by π from E^n is orientable over $\tilde{V}$. ($\tilde{V}$ is constructed so that its fundamental group is the largest subgroup of $\pi_1(V)$ which preserves the orientation of E^u over V. See Williams [7] again.) $\tilde{V}$ is chosen so that the fibers of $\tilde{E}^u$ over the two points in a fiber of $\pi: \tilde{V} \to V$ have the opposite orientation.

Next define the quotient space $W = \tilde{V} \cup \overline{U}_0/\sim$ where $\sim$ identifies x, y and $\pi(x) = \pi(y)$ if x and y lie in the same fiber of $\tilde{V}$ over $\partial U \cap \partial U_0$. Define the map $T: W \to W$ by $T|\overline{U}_0 =$ identity and $T|\tilde{V}$ is the map which interchanges the two points in a fiber of π. f lifts to a map $\tilde{f}: W \to W$ defined by

(1) $\tilde{f}|\overline{U}_0 = f|\overline{U}_0$, and

(2) $\tilde{f}: \tilde{V} \to W$ is the map covering $f: V \to W$ which preserves the orientation of $\tilde{E}^n$.

f also lifts to the map $T\tilde{f}$. Note that if $p \in \Omega_0$ is a fixed point of f^i, then $L(p) = (-1)^u$ for $\tilde{f}^i$ and $L(p) = (-1)^{u+1}$ for $T\tilde{f}^i$ by the proposition stated above. We now apply the Lefschetz trace formula to $\tilde{f}^i$ and $T\tilde{f}^i$:

$$(*) \quad \sum_{p \in \text{fix}\, \tilde{f}^i} L(p) - \sum_{p \in \text{fix}\, T\tilde{f}^i} L(p) = \sum_j (-1)^j \operatorname{Tr} \tilde{f}^i_{*j} - \sum_j (-1)^j \operatorname{Tr} T\tilde{f}^i_{*j}.$$

By observing that $(T\tilde{f}^i_*) = T_*(\tilde{f}^i_*)$ we conclude that there are integral matrices A and B such that the right-hand side of (*) is equal to $\operatorname{Tr} A^i - \operatorname{Tr} B^i$ for all i. Again Williams [7] has more detail. On the other hand, since $\tilde{f}^i|U_0 = T\tilde{f}^i|U_0$, the left side of (*) is equal to $(-1)^u$ (# fixed points of $\tilde{f}^i$ + # fixed points of $T\tilde{f}^i$). Over each fixed point of f^i in Ω_0, exactly one of $\tilde{f}^i$ and $T\tilde{f}^i$ has two fixed points. Therefore

(# fixed points of $f^i|\Omega_0) = (-1/2)^u$ (# fixed points of $\tilde{f}^i|\tilde{V}$ + # fixed points of $T\tilde{f}^i|\tilde{V}$). Hence there are integral matrices A and B such that

$$(\#\text{ fixed points of } f^i \mid \Omega_0) = \operatorname{Tr} A^i - \operatorname{Tr} B^i,$$

proving that the zeta function of $f|\Omega_0$ is rational. Because the spectral decomposition of $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$ into topologically transitive pieces is finite, and because $\zeta_f(t) = \prod_i \zeta_f|\Omega_i(t)$, this suffices to prove the theorem.

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