ON THE EXTENSION OF LIPSCHITZ, LIPSCHITZ-HÖLDER CONTINUOUS, AND MONOTONE FUNCTIONS[1]

BY GEORGE J. MINTY

Communicated by Gian-Carlo Rota, September 22, 1969

1. **Introduction.** The well-known theorem of Kirszbraun [9], [14] asserts that a Lipschitz function from R^n to itself, with domain a finite point-set, can be extended to a larger domain including any arbitrarily chosen point. (The Euclidean norm is essential; see Schönbeck [16], Grünbaum [8].) This theorem was rediscovered by Valentine [17] using different methods. The writer [12] proved the same fact for a "monotone" function, and Grünbaum [9] combined these two theorems into one. A further improvement to the writer's theorem was given by Debrunner and Flor [6], who showed that the desired new functional value could always be chosen in the convex hull of the given functional values; several different proofs of this fact have now been given (see [14], [3]). An easy consequence of Kirszbraun's theorem is that a Lipschitz function in Hilbert space with maximal domain is everywhere-defined (see [11], [13]).

It was shown by S. Banach [1] that a real-valued function defined on a subset of a metric space and satisfying $|f(y_1)-f(y_2)| \leqq [\delta(y_1, y_2)]^\alpha$, with $0<\alpha\leqq 1$ (we call this "Lipschitz-Hölder continuity"), can be extended to the whole metric space so as to satisfy the same inequality. Banach's theorem was rediscovered by Czipszer and Gehér [4] in case $\alpha=1$ (but note that Banach's result follows, since $[\delta(y_1, y_2)]^\alpha$ is another metric if $\alpha\leqq 1$). For a general review of the above subjects, see the article of Danzer, Grünbaum, and Klee [5]; see also [7].

In this paper, we give a unified method for proving all the above results, and also new theorems, the most striking of which is the following generalization of the Kirszbraun and Banach theorems:

THEOREM 1. *Let H be a Hilbert space, M a metric space, $D\subset M$. Suppose $f: D\to H$ satisfies $\|f(y_1)-f(y_2)\| \leqq [\delta(y_1, y_2)]^\alpha$ $(0<\alpha\leqq 1)$. Then there exists an extension of f to all of M satisfying the same inequality, if either*

(i) $\alpha\leqq\frac{1}{2}$, *or*

AMS Subject Classifications. Primary 2670; Secondary 5234.

Key Words and Phrases. Kirszbraun's Theorem, Hölder-continuous functions, extension theorems, Lip (α).

[1] This research supported by National Science Foundation Grant GP-11878.

(ii) *M is an inner product space, with metric given by* $k^{1/\alpha}\|y_1-y_2\|$, *where* $k>0$.

Moreover, the extension can be performed so that the range of the extension lies in the closed convex hull of the range of f; thus

$$|||f|||_\alpha = \sup_y \|f(y)\| + \sup_{y_1\neq y_2} \frac{\|f(y_1) - f(y_2)\|}{[\delta(y_1, y_2)]^\alpha}$$

is not increased.

(Note that in case (ii), the inequality reads $\|f(y_1)-f(y_2)\| \leqq k\|y_1-y_2\|^\alpha$. The important point is that k need not be changed when the extension is performed.) To the best of the writer's knowledge, no theorems on extension of Hölder-continuous functions with infinite-dimensional range have been known until now, and the present theorem is new even for finite-dimensional Hilbert space.

2. **Main theorem.** Let X be a vector space over the real numbers. A real-valued function on X is called *finitely lower semicontinuous* if its restriction to any finite-dimensional subspace of X is lower semicontinuous, the subspace being taken with the "usual" topology. (Examples are: a linear function, a quadratic form; neither need be "bounded".) Now let Y also be a space. A function $\Phi: X\times Y\times Y\to R$, written $\Phi(x, y_1, y_2)$, shall be called a *Kirszbraun function* (*K-function*) provided: (1^0) for each fixed y_1, y_2 it is a finitely lower semicontinuous, convex function of x; and (2^0) for any sequence $(x_1, y_1), \cdots, (x_m, y_m)$ in $X\times Y$, any $y\in Y$, and any probability vector $(\mu_1, \cdots, \mu_m)$, we have

$$\sum_{i,j}^{m} \mu_i\mu_j\Phi(x_i - x_j, y_i, y_j) \geqq 2 \sum_{i}^{m} \mu_i\Phi(x_i - x, y_i, y) \tag{2.1}$$

where x stands for $\sum_j^m \mu_j x_j$.

If X is a finite-dimensional space, we shall call Φ a *finite-dimensional K-function* if it satisfies the above definition with m replaced by $1+\dim X$.

THEOREM 2 (MAIN THEOREM). (A) *Let X and Y be as above, and Φ be a K-function. Let $(x_1, y_1), \cdots, (x_m, y_m)$ be a sequence in $X\times Y$ such that $\Phi(x_i-x_j, y_i, y_j)\leqq 0$ for all i, j, and let y be any element of Y. Then there exists a vector x such that $\Phi(x_i-x, y_i, y)\leqq 0$ for all i. Furthermore, x can be chosen in the convex hull of $\{x_1, \cdots, x_m\}$.*

(B) *The same statement holds if X is finite-dimensional, and Φ is a corresponding finite-dimensional K-function.*

PROOF. (A) Let P_m be the set of probability-vectors in R^m. Consider $\Phi: P_m \times P_m \to R$, defined as $\Phi(\mu, \lambda) = \sum_i \mu_i \phi(x_i - x, y_i, y)$ where x stands for $\sum_j \lambda_j x_j$. Now, P_m is compact; also, Φ is convex and lower semicontinuous in λ and concave and upper semicontinuous in μ. Thus, by von Neumann's Minimax Theorem [2] there exists a pair (μ^0, λ^0) in $P_m \times P_m$ such that for all (μ, λ) in $P_m \times P_m$

$$\Phi(\mu^0, \lambda) \geqq \Phi(\mu, \lambda^0). \tag{2.2}$$

By putting $\lambda = \mu^0$, we see that the left-hand side of (2.2) is nonpositive; by putting μ a Kronecker delta on the right, we have the conclusion.

(B) First apply Helly's Theorem (see [2]) to reduce the case of general m to the case $m = n+1$; then apply the proof of (A) with $m = n+1$.

3. **Examples of K-functions.** It is easily verified that the following are K-functions: a negative (constant) real number, a linear form in x, a positive semidefinite quadratic form in x.

For any space Y and $\delta: Y \times Y \to R$ such that $\delta(y_1, y_2) \geqq 0$ and $\delta(y_1, y_3) \leqq \delta(y_1, y_2) + \delta(y_3, y_2)$, then $(-\delta)$ is a K-function. In particular, δ might be a metric on Y.

In case Y is a space with an operation "minus" satisfying $(y_1 - y_3) - (y_2 - y_3) = y_1 - y_2$ (for example, a group, with $y_1 - y_2 = y_1 y^{-1}$), and $\psi: X \times Y \to R$ satisfies

$$\sum_{i,j} \mu_i \mu_j \psi(x_i - x_j, y_i - y_j) \geqq 2 \sum_i \mu_i \psi(x_i - x, y_i) \tag{3.1}$$

then $\Phi(x, y_1, y_2) = \psi(x, y_1 - y_2)$ satisfies the inequality of the definition of "K-function." If Y is a linear space, then ψ might be a negative semidefinite quadratic form in y, or a bilinear form in x and y; these give rise to K-functions.

If x is the real numbers, then x^4 is a K-function; this follows from the identity

$$\sum \mu_i \mu_j \left| x_i - x_j \right|^4 = 2 \sum_i \mu_i \left| x_i - x \right|^4 + 6\left(\sum_i \mu_i x_i^2 - x^2 \right)^2$$

(where x is $\sum_i \mu_i x_i$, as before, and $\sum_i \mu_i = 1$).

Moreover, *any linear combination of K-functions with nonnegative coefficients is a K-function.* (Of course, assuming X, Y the same for all of them.)

COROLLARIES TO THEOREM 1. Kirszbraun's Theorem follows from the case $\psi(x, y)=\|x\|^2-\|y\|^2$. The Debrunner-Flor Lemma mentioned in the Introduction is the case where $\psi(x, y)$ is a bilinear form. The theorem of Grünbaum [9] is contained in the case $\psi=k_1(\|x\|^2-\|y\|^2)+k_2\langle x, y\rangle$, with nonnegative k_1, k_2.

Letting X be a Hilbert space and Y a metric space, and taking $\Phi(x, y_1, y_2)=\|x\|^2-\delta(y_1, y_2)$, we obtain the necessary lemma to prove part (ii) of Theorem 1, with $\alpha=\frac{1}{2}$. The proof parallels closely the usual proof of the extension theorem for Lipschitz functions (see [11] or [13]), slightly modified to keep the range of the extension in the closed convex hull of the range of f.

As remarked in the Introduction, $[\delta(y_1, y_2)]^\beta$ is also a metric if $\beta\leqq 1$; hence we have an extension theorem for f satisfying $\|f(y_1)-f(\lambda_2)\|\leqq[\delta(y_1, y_2)]^\alpha$ with $\alpha\leqq\frac{1}{2}$. Indeed, if $g(\gamma)$ is a real-valued function of $\gamma\geqq 0$ with $g(0)=0$, $g(\gamma)>0$ for $\gamma>0$, g nondecreasing in γ, and $\gamma^{-1}g(\gamma)$ nonincreasing for $\gamma>0$, we have (for $\gamma_1, \gamma_2>0$):

$$\gamma_1 g(\gamma_1+\gamma_2)\leqq(\gamma_1+\gamma_2)g(\gamma_1),$$
$$\gamma_2 g(\gamma_1+\gamma_2)\leqq(\gamma_1+\gamma_2)g(\gamma_2)$$

whence (by adding) g is subadditive, so that $g\circ\delta$ is again a metric. Thus $g(\gamma)=\gamma^\beta$, with $\alpha\leqq 1$, is a special case.

It has recently been established by H. Brězis and C. M. Fox that $\psi(x, y)=-\|y\|^\beta$ is a K-function for $0<\beta\leqq 2$ in a Euclidean space (or an inner product space). Brězis uses M. Riesz' Convexity Theorem; Fox gives an elementary (but ingenious) proof of the stronger statement

$$\sum_{i,j}^{m}\mu_i\mu_j\|y_i-y_j\|^{2\alpha}\leqq\sum_{i,j}^{m}\mu_i\mu_j(\|y_i\|^2+\|y_j\|^2)^\alpha\quad(\text{for } 0<\alpha\leqq 1).\tag{3.2}$$

J. Moser and the writer have simplified Fox's proof, as follows:

LEMMA. *For $x_1, \cdots, x_m$ in an inner product space, and $a_1, \cdots, a_m>0$, $\beta>0$, note*

$$\sum_{i,j}\frac{\langle x_i, x_j\rangle}{(a_i+a_j)^\beta}=\frac{1}{\Gamma(\beta)}\int_0^\infty\left\|\sum_i e^{-a_it}x_i\right\|^2 t^{\beta-1}dt\tag{3.3}$$

and thus it is nonnegative.

Now write the left-hand side of (3.2) as

$$\sum_{i,j}^{m}\mu_i\mu_j(\|y_i\|^2+\|y_j\|^2)^\alpha\left[1-\frac{2\langle y_i, y_j\rangle}{\|y_i\|^2+\|y_j\|^2}\right]^\alpha,$$

apply Bernoulli's inequality to the expression in square brackets, and then the lemma, with $x_i = \mu_i y_i$, and $a_i = \|y_i\|^2$. (The case where some y_i are zero is easily disposed of by a continuity argument.)

The above argument is easily generalized to show $-[Q(y_1 - y_2)]^\alpha$, with $0 < \alpha \leqq 1$, is a K-function if Q is a positive semidefinite quadratic form in a linear space Y. Part (ii) of Theorem 1 is proved by use of the K-function $\|x\|^2 - k^2\|y_1 - y_2\|^{2\alpha}$, followed by the "usual" argument for Lipschitz functions.

J. Moser and G. Schober have shown that if X is one-dimensional, then $-[\delta(y_1, y_2)]^2$ is a finite-dimensional K-function; i.e., it satisfies the desired inequality with $m = 2$. Schober's proof considers separately the case $\delta(y_1, y_2)^2 \leqq \delta(y_1, y)^2 + \delta(y_2, y)^2$ which is easy, and the opposite case, which is treated by the standard maximization argument of differential calculus applied to the function $f(\mu) = \mu(1-\mu)\delta(y_1, y_2)^2 - \mu\delta(y_1, y)^2 - (1-\mu)\delta(y_2, y)^2$. The extension theorem of Banach follows by Theorem 2, part (B), applied to $|x|^2 - [\delta(y_1, y_2)]^2$.

NOTE ADDED IN PROOF. Banach's theorem mentioned above is more probably due to McShane (Bull. Amer. Math. Soc. **40** (1934), 837–842). (2°) The hypothesis "finitely lower-semicontinuous" follows from the other hypotheses of the definition of "K-function", and so can be dropped. (3°) Hayden, Wells, and Williams of the University of Kentucky have generalized the extension-theorem to cover functions from one L^p-space to another (unpublished work).

REFERENCES

1. S. Banach, *Introduction to the theory of real functions*, Monografie Mat., Tom 17, PWN, Warsaw, 1951. MR **13**, 216.

2. C. Berge and A. Ghouila-Houri, *Programmes jeux et rêseaux de transport*, Dunod, Paris, 1962; English transl., Methuen, London and Wiley, New York, 1965. MR **33** #1137; MR **33** #7114.

3. F. E. Browder, *Existence and perturbation theorems for nonlinear maximal monotone operators in Banach spaces*, Bull. Amer. Math. Soc. **73** (1967), 322–327. MR **35** #3495.

4. J. Czipszer and L. Gehér, *Extension of functions satisfying a Lipschitz condition*, Acta Math. Acad. Sci. Hungar. **6** (1955), 213–220. MR **17**, 136.

5. L. Danzer, B. Grünbaum and V. Klee, *Helly's theorem and its relatives*, Proc. Sympos. Pure Math., vol. **7**, Amer. Math. Soc., Providence, R. I., 1963, pp. 101–180. MR **28** #524.

6. H. Debrunner and P. Flor, *Ein Erweiterungssatz für monotone Mengen*, Arch. Math. **15** (1964), 445–447. MR **30** #428.

7. D. G. de Figueiredo and L. A. Karlovitz, *On the extension of contractions on normed spaces*, University of Maryland Technical Note, BN-563.

8. B. Grünbaum, *On a theorem of Kirszbraun*, Bull. Res. Council Israel Sect. F **7** (1957/58), 129–132. MR **21** #5155.

9. ———, *A Generalization of theorems of Kirszbraun and Minty*, Proc. Amer. Math. Soc. **13** (1962), 812–814. MR **27** #6110.

10. M. D. Kirszbraun, *Über die zusammenziehenden and Lipschitzschen Transformationen*, Fund. Math. 22 (1934), 7–10.

11. E. J. Mickle, *On the extension of a transformation*, Bull. Amer. Math. Soc. **55** (1949), 160–164. MR **10**, 691.

12. G. J. Minty, *On the simultaneous solution of a certain system of linear inequalities*, Proc. Amer. Math. Soc. **13** (1962), 11–12. MR **26** #573.

13. ———, *Monotone (nonlinear) operators in Hilbert space*, Duke Math. J. **29** (1962), 341–346. MR 29 #6319.

14. ———, *On the generalization of a direct method of the calculus of variations*, Bull. Amer. Math. Soc. **73** (1967), 315–321. MR **35** #3501.

15. I. J. Schoenberg, *On a theorem of Kirszbraun and Valentine*, Amer. Math. Monthly **60** (1953), 620–622. MR **15**, 341.

16. S. O. Schönbeck, *Extension of nonlinear contractions*, Bull. Amer. Math. Soc. **72** (1966), 99–101. MR **37** #1960.

17. F. A. Valentine, *A Lipschitz condition preserving extension for a vector function*, Amer. J. Math. **67** (1945), 83–93. MR **6**, 203.

INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47401