ABELIAN QUOTIENTS OF THE MAPPING CLASS GROUP OF A 2-MANIFOLD

BY JOAN S. BIRMAN

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Let T_g be a closed, orientable 2-manifold of genus g, and let M_g be the mapping class group of T_g, that is the group of orientation-preserving homeomorphisms of $T_g \rightarrow T_g$ modulo those isotopic to the identity. The following theorem was proved by D. Mumford in [6]: If $[M_g, M_g]$ is the commutator subgroup of M_g, then $A_g = M_g/[M_g, M_g]$ is a finite cyclic group whose order is a divisor of 10. We give a very brief and elementary reproof of Mumford's theorem, and at the same time improve his result to show that the order of A_g is 2 if $g \geqq 3$.

Generators for M_g are well known, and a particularly convenient set is given by W. B. R. Lickorish in [3]. Lickorish's generators are "screw maps" about closed curves on the surface T_g (the definition of a screw map is the same as that in [6]), and Lickorish shows that the screw maps about the curves $\{u_i, z_i, c_j; 1 \leqq i \leqq g, 1 \leqq j \leqq g-1\}$ in Figure 1 generate M_g.

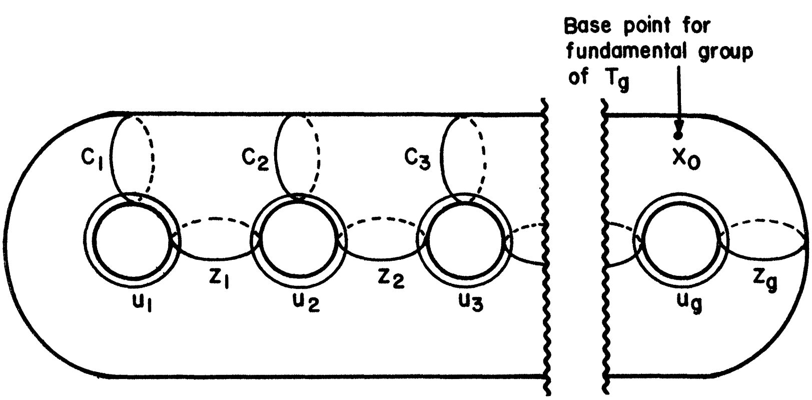

FIGURE 1

By a well-known result [5] the group M_g is isomorphic to a group of automorphism classes (cosets of the subgroup of inner automorphisms in the group of all automorphisms) of the fundamental group

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$\pi_1 T_g$ of the surface T_g. Choosing generators $\{t_i, s_i;\ 1 \leqq i \leqq g\}$ for $\pi_1 T_g$ as illustrated in Figure 2, and denoting screw maps about the curves u_i, z_i and c_i by U_i, Z_i and C_i respectively, the automorphisms of $\pi_1 T_g$ corresponding to Lickorish's generators of M_g are easily determined (see [1]), and are given explicitly as follows:

(1) $$U_i\colon t_i \to t_i s_i \qquad 1 \leqq i \leqq g$$

$$Z_i\colon s_i \to t_i^{-1} t_{i+1} s_i \qquad 1 \leqq i \leqq g-1$$

(2) $$s_{i+1} \to s_{i+1} t_{i+1}^{-1} t_i \qquad 1 \leqq i \leqq g-1$$

$$Z_g\colon s_g \to t_g^{-1} s_g$$

$$C_i\colon s_j \to t_i s_j t_i^{-1} \qquad j < i$$

(3) $$t_j \to t_i t_j t_i^{-1} \qquad j < i \qquad 1 \leqq i \leqq g-1$$

$$s_i \to s_i t_i^{-1}$$

where it is understood that every generator of $\pi_1 T_g$ which is not listed explicitly is unaltered by the screw maps.

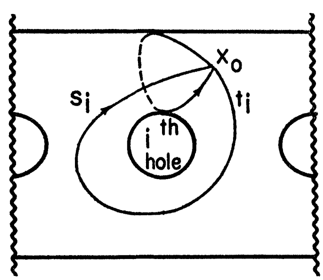

FIGURE 2

This representation of M_g as a group of automorphism classes provides a very simple tool for calculation in M_g. If one suspects that two sequences of screw maps are equivalent in M_g, one simply calculates the induced automorphisms, and determines if they agree modulo an inner automorphism. Using this procedure, the following relations can be verified to hold in M_g:

(4) $$U_i Z_i U_i = Z_i U_i Z_i \qquad 1 \leqq i \leqq g$$

(5) $$U_{i+1} Z_i U_{i+1} = Z_i U_{i+1} Z_i \qquad 1 \leqq i \leqq g-1$$

(6) $C_iU_iC_i = U_iC_iU_i \qquad 1 \leqq i \leqq g$

(7) $(C_1U_1Z_1U_2Z_2 \cdots U_gZ_g^2U_g \cdots Z_2U_2Z_1U_1C_1)^2 = 1$

(8) $(C_1U_1Z_1U_2Z_2 \cdots U_gZ_g)^{2g+1} = 1$

(9) $(U_1Z_1U_2C_2)^5 = (C_1U_1Z_1U_2C_2^2U_2Z_1U_1C_1) \qquad \text{if } g \geqq 3$

(10) $(U_1Z_1U_2Z_2U_3C_3)^7 = (C_1U_1Z_1U_2Z_2U_3C_3^2U_3Z_2U_2Z_1U_1C_1) \quad \text{if } g \geqq 4.$

Relations (4)–(8) above were determined by the author in [1]; relations (9) and (10) are new, to the author's knowledge.

We now consider the abelianizing homomorphism $\alpha: M_g \rightarrow A_g$. Under α, relation (4) goes over to

$$\alpha(U_i)\alpha(Z_i)\alpha(U_i) = \alpha(Z_i)\alpha(U_i)\alpha(Z_i). \tag{11}$$

Since all elements in A_g commute, (11) implies:

$$\alpha(U_i) = \alpha(Z_i). \tag{12}$$

Since similar relations link the entire set of generators of M_g, we obtain immediately that A_g is a cyclic group. Denoting the single generator of A_g by $h=\alpha(U_i)$, equations (7), (8), (9) and (10) then give

$$\begin{aligned} &h^{(2g+1)(4)} = h^{(2g+1)(2g+2)} = 1 \qquad \text{for all } g, \\ &h^{10} = 1 \qquad \text{if } g \geqq 3, \\ &h^{28} = 1 \qquad \text{if } g \geqq 4. \end{aligned} \tag{13}$$

Together these imply that the order of h is a divisor of 10 if $g=2$, while for $g \geqq 3$ the order of h divides 2.

It only remains to prove that the order of A_g cannot be 1. To establish this, we make use of the well-known fact that the group $\mathrm{Sp}(2g, Z)$ of $2g$-by-$2g$ symplectic matrices with integral entries is a quotient group of M_g [5], and hence the commutator quotient group of $\mathrm{Sp}(2g, Z)$ is a quotient group of A_g. The author is grateful to J. Mennicke for pointing out that it follows from known work [2] that the commutator quotient group of $\mathrm{Sp}(2g, Z)$ is of order 2; hence A_g is of order 2 for all $g \geqq 3$. For $g=2$ it is known that A_g is cyclic of order 10.

Some geometric insight into the proof outlined above is obtained by noting that the cyclic nature of A_g is an immediate consequence of relations (4), (5), (6). For the case of the torus ($g=1$) these reduce to the single relation:

$$U_1Z_1U_1 = Z_1U_1Z_1$$

which is classical. Now, it is easily established that this relation remains valid on a torus with n points removed. Since all pairs (U_1, Z_i), (U_i, C_i) and (Z_i, U_{i+1}) of generators of M_g can be displayed as appropriate pairs of screw maps on subsets of T_g which are homeomorphic to a punctured torus, relations (4), (5), and (6) are seen to follow directly from the corresponding relation in M_1. The order of the single generator of A_g is determined by relations (7), (8), (9), and (10). Of these, relations (7) and (8) basically express symmetries in the geometric realization of the surface T_g; relations (9) and (10) are obtained from (7) and (8) specialized to the cases $g=2$ and 3 respectively, and carried over to subsets of T_g which are homeomorphic to (T_2-one point) and (T_3-one point) respectively.

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Stevens Institute of Technology, Hoboken, New Jersey 07030