MONOTONE OPERATORS AND NONLINEAR INTEGRAL EQUATIONS OF HAMMERSTEIN TYPE

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A nonlinear integral equation of Hammerstein type is one of the form

$$u(x) + \int_G K(x, y)f(y, u(y))dy = 0, \tag{1}$$

where G is a measure space with σ-finite measure dy and the unknown function $u(x)$ is defined on G. In operator-theoretic terms, the problem of determining the solutions of the equation (1) with u lying in a given Banach space Y of functions on G can be put in the form of the nonlinear functional equation

$$u + AN(u) = 0 \tag{2}$$

with the linear and nonlinear mappings A and N given by

$$Av(x) = \int_G K(x, y)v(y)dy, \qquad Nu(x) = f(x, u(x)). \tag{3}$$

In the present note, we establish general results on the existence and uniqueness of solutions of equation (2) for the Banach space $Y = X^*$ under appropriate assumptions of weak monotonicity type upon the mappings A and N. We note that Hammerstein equations have an extensive literature which includes Hammerstein [11], Iglisch [12], Golomb [10], Dolph [7], Rothe [18], Vainberg [19], [20], and Krasnosel'skiĭ [16]. The first application of the concept of monotone operator in this problem was made implicitly by Golomb [10] and explicitly by Vainberg [19]. More recent papers applying monotonicity concepts to Hammerstein equations include Dolph-Minty [8], Kolodner [13], Brézis [3], Kolomy [14], [15], Amann [1], [2], de Figueiredo-Gupta [9] and Vainberg [20].

We employ the following definitions: If X is a real Banach space, X^* its conjugate space, we let (w, u) denote the duality pairing between the element w of X^* and the element u of X. A mapping A of X into X^* is said to be monotone if for all u, v in X we have

$$(A(u) - A(v), u - v) \geqq 0.$$

A mapping N of X^* into X is said to be hemicontinuous if it is continuous from each line segment of X^* to the weak topology of X.

DEFINITION 1. If A is a bounded monotone linear mapping of X into X^*, then A is said to be *angle-bounded* with constant $c \geqq 0$ if for all u, v in X

$$| (A(u), v) - (A(v), u) | \leqq 2c\{(A(u), u)\}^{1/2}\{(A(v), v)\}^{1/2}.$$

DEFINITION 2. If A is a bounded linear mapping of X into X^*, A is said to be *symmetric* if for all u and v in X,

$$(A(u), v) = (A(v), u).$$

THEOREM 1. *Let X be a general real Banach space, X^* its conjugate space. Let A be a monotone angle-bounded continuous linear mapping of X into X^* with constant of angle-boundedness $c \geqq 0$. Let N be a hemicontinuous (possibly nonlinear) mapping of X^* into X such that for a given constant $k \geqq 0$*

$$(v - v_1, N(v) - N(v_1)) \geqq -k\|v - v_1\|_{X^*}^2 \tag{4}$$

for all v and v_1 in X^*. Suppose finally that there exists a constant R with $k(1+c^2)R<1$ such that for all u in X

$$(A(u), u) \leqq R\|u\|_X^2. \tag{5}$$

Then there exists exactly one solution w in X^* of the nonlinear equation

$$w + AN(w) = 0. \tag{6}$$

Some special cases of Theorem 1 are the following:

THEOREM 2. *Let X be a general real Banach space, X^* its conjugate space, A a bounded linear mapping of X into X^* with A monotone and angle-bounded. Let N be a hemicontinuous (possibly nonlinear) mapping of X^* into X which is monotone, that is,*

$$(v - v_1, N(v) - N(v_1)) \geqq 0$$

for all v and v_1 in X^*.

Then there exist exactly one solution w in X^* of the equation

$$w + AN(w) = 0.$$

The result of Theorem 2 was obtained by Amann [1] under an assumption that X^* has a dense continuous linear imbedding in a Hilbert space. As we show below, such assumptions are not needed in the proof of Theorem 1, nor in the proof of Theorem 2 which is a special case of Theorem 1 for $k=0$. Another special case is the following

THEOREM 3. *Let X be a general real Banach space, X^* its conjugate space, A a bounded linear mapping of X into X^* which is monotone and symmetric. Suppose that N is a hemicontinuous (possibly nonlinear) mapping of X^* into X such that for a given $k \geqq 0$ and all v, v_1 in X^*,*

$$(v - v_1, N(v) - N(v_1)) \geqq -k\|v - v_1\|_{X^*}^2.$$

Suppose that $k\|A\| < 1$.

Then the equation $w + AN(w) = 0$ has exactly one solution w in X^*.

We note that when A is symmetric then A is angle-bounded with constant of angle-boundedness $c = 0$.

The result of Theorem 3 was obtained by Golomb [10] for $X = L^2(G)$ and by Vainberg [20] for $X = L^p(G)$, using variational methods. Our method, on the other hand, consists in splitting the linear operator A via a Hilbert space H (Theorem 4) and reducing the equation (6) to an equivalent equation in H, which is then solved by using the results of Browder [5] and Minty [17] for monotone operator equations.

We may add that the proof of Theorems 1, 2, 3 and 4 can be adapted in a straightforward manner to the case of a real locally-convex space.

We turn now to the proof of Theorem 1. An essential tool in that proof is the following auxiliary theorem.

THEOREM 4. *Let X be a Banach space, X^* its conjugate space, A a bounded linear mapping of X into X^* which is monotone and angle-bounded. Then there exists a Hilbert space H, a continuous linear mapping S of X into H with S^* injective and a bounded skew-symmetric linear mapping B of H into H such that $A = S^*(I+B)S$, and the following two inequalities hold:*

(i) $\|B\| \leqq c$, *with c the constant of angle-boundedness of A.*

(ii) $\|S\|^2 \leqq R$ *if and only if for all u in X, $(A(u), u) \leqq R\|u\|_X^2$.*

PROOF. We introduce a symmetric bilinear form on X by setting

$$[u, u_1] = \tfrac{1}{2}\{(A(u), u_1) + (A(u_1), u)\}.$$

Since $[u, u] = (A(u), u) \geqq 0$ by the monotonicity of A, we have the Cauchy-Schwarz inequality

$$|[u, u_1]| \leqq \{[u, u]\}^{1/2}\{[u_1, u_1]\}^{1/2}. \tag{7}$$

Let N be the subset of those u in X for which $[u, u] = 0$. It follows from (7) that $N = \{u \mid u \text{ in } X,\ [u, v] = 0 \text{ for all } v \text{ in } X\}$ and hence, N is a closed vector-subspace of X. Let H_0 denote the quotient vector-space

X/N and endow H_0 with the inner product $[u+N, u_1+N]=[u, u_1]$. Under this inner product, H_0 is a pre-Hilbert space. We denote by H the Hilbert space obtained from H_0 by completing it with respect to this inner product. We let S denote the linear mapping of X into H obtained from the natural projection of X onto X/N. Since A is a bounded linear mapping of X into X^*, there exists a least constant $R \geqq 0$ such that $(A(u), u) \leqq R\|u\|_X^2$ for all u in X. Hence, for all u in X we have

$$\|S(u)\|_H^2 = [S(u), S(u)] = (A(u), u) \leqq R\|u\|_X^2$$

that is, S is a bounded linear mapping of X into H with $\|S\|^2 \leqq R$. Since the range of S coincides with H_0, S has a dense range in H. Hence the adjoint mapping S^* of H into X^* is injective.

Since A is angle-bounded with constant of angle-boundedness $c \geqq 0$, it follows that for all u, v in X

$$|(A(u), v) - (A(v), u)| \leqq 2c\|S(u)\|_H\|S(v)\|_H.$$

Hence the function

$$h(S(u), S(v)) = \tfrac{1}{2}\{(A(u), v) - (A(v), u)\}$$

is well defined on H_0 and is a bounded bilinear form on H_0. Let h also denote the unique extension of this bounded bilinear form on H_0 to H. It follows that there exists a well-defined and unique bounded linear mapping B of H into H such that for all u, v in X

$$h(S(u), S(v)) = [B(S(u)), S(v)].$$

Since $h(S(u), S(v)) = -h(S(v), S(u))$ for all u, v in X, it follows that B is skew-symmetric on H_0 and hence on all of H by continuity. We note that

$$|[B(S(u)), S(v)]| \leqq c\|S(u)\|_H\|S(v)\|_H$$

for all u, v in X and so $\|B\| \leqq c$.

Finally for all u, v in X

$$\begin{aligned}(A(u), v) &= [u, v] + h(S(u), S(v)) = [S(u), S(v)] + [B(S(u)), S(v)] \\ &= [(I + B)S(u), S(v)] = (S^*(I + B)S(u), v).\end{aligned}$$

Thus $A = S^*(I+B)S$. This completes the proof of the theorem.

We also need the following elementary lemma in the proof of Theorem 1.

LEMMA. *Let H be a given Hilbert space, B a skew-symmetric bounded linear mapping of H into H. Then the bounded linear mapping $I+B$ is* *a monotone bijective mapping of H onto H. Further, for any u in H we have*

$$[(I+B)^{-1}(u), u] \geqq \frac{1}{1+\|B\|^2}\|u\|_H^2.$$

PROOF OF THEOREM 1. Suppose w in X^* is a solution of the equation (6). By Theorem 4, $A=S^*(I+B)\,S$ and equation (6) becomes

$$w + S^*(I+B)SN(w) = 0. \tag{8}$$

Since S^* is injective, there is a unique u in H such that $w=S^*(u)$ and equation (8) becomes

$$S^*(u) + S^*(I+B)SNS^*(u) = 0, \tag{9}$$

i.e.,

$$S^*(u + (I+B)SNS^*(u)) = 0. \tag{10}$$

Since S^* is injective, equation (10) is equivalent to

$$u + (I+B)SNS^*(u) = 0. \tag{11}$$

Hence equation (6) has exactly one solution in X^* if and only if equation (11) has exactly one solution in H. Now, by the lemma, equation (11) is equivalent to the equation

$$(I+B)^{-1}(u) + SNS^*(u) = 0. \tag{12}$$

Let $T=(I+B)^{-1}+SNS^*$. For u, v in H,

$$[T(u)-T(v), u-v] = [(I+B)^{-1}(u-v), u-v] + [SNS^*(u) - SNS^*(v), u-v].$$

By the lemma and Theorem 4(i)

$$[(I+B)^{-1}(u-v), u-v] \geqq \frac{1}{1+\|B\|^2}\|u-v\|_H^2 \geqq \frac{1}{1+c^2}\|u-v\|_H^2.$$

On the other hand,

$$[SNS^*(u)-SNS^*(v), u-v] = (S^*(u)-S^*(v), NS^*(u)-NS^*(v)) \geqq -k\|S^*(u)-S^*(v)\|_{X^*}^2 \geqq -kR\|u-v\|_H^2$$

by Theorem 4(ii). Combining these inequalities, we see that

$$[T(u)-T(v), u-v] \geqq (1/(1+c^2) - kR)\|u-v\|_H^2 \geqq c_1\|u-v\|_H^2$$

where $c_1=1/(1+c^2)-kR>0$ since $k(1+c^2)R<1$ by hypothesis.

Thus T is a monotone hemicontinuous mapping of H into H and T is injective. Moreover, for u in H we have

$$[T(u), u] = [T(u) - T(0), u - 0] + [T(0), u]$$
$$\geqq c_1\|u\|_H^2 - \|T(0)\|_H\|u\|_H.$$

Since $[T(u), u]/\|u\|_H \to \infty$ as $\|u\|_H \to \infty$, T is coercive. It then follows from the results of Browder [5] and Minty [17] that T maps H onto H injectively. Hence (12) has exactly one solution in H and so by our preceding discussion, equation (6) has exactly one solution in X^*. q.e.d.

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