

THE ACTION OF A REAL SEMISIMPLE GROUP ON A COMPLEX FLAG MANIFOLD. I: ORBIT STRUCTURE AND HOLOMORPHIC ARC COMPONENTS¹

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1. Introduction. This paper describes a topic that is of some interest in Lie groups and in differential and algebraic geometry. The topic shows good promise of being the “correct” context for explicit realization of those series of irreducible unitary representations of semisimple Lie groups that come into the Plancherel formula, so it probably is also of interest in harmonic analysis. We start with an example.

Let X be the Riemann sphere, viewed as $\mathbf{C} \cup \{\infty\}$ via stereographic projection. Then the group G of all holomorphic automorphisms of X consists of the linear fractional transformations

$$\pm \begin{pmatrix} a & c \\ b & d \end{pmatrix} : z \rightarrow \frac{az + b}{cz + d}, \quad \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = 1;$$

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that expresses G as a complex matrix group $\mathbf{SL}(2, \mathbf{C})/\{\pm I\}$. Let $G_0 = \mathbf{SL}(2, \mathbf{R})/\{\pm I\}$, subgroup of elements represented by real matrices, so G is the complexification of G_0 . The complex group G is transitive on X but the real group G_0 is not. In fact X is the union of two open G_0 -orbits

$$G_0(\sqrt{1}) = \{z \in \mathbf{C} : \text{Im } z > 0\} \quad \text{and} \quad G_0(-\sqrt{-1}) = \{z \in \mathbf{C} : \text{Im } z < 0\}$$

and the closed G_0 -orbit

$$G_0(0) = \{z \in \mathbf{C} : \text{Im } z = 0\} \cup \{\infty\}.$$

The Cayley transform

$$c = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix} \in G$$

of X relates these orbits, so they are

$$G_0(x_0) \quad \text{and} \quad G_0(c^2x_0) \text{ open,} \quad G_0(cx_0) \text{ closed,} \quad x_0 = \sqrt{-1}.$$

The series of irreducible unitary representations of G_0 are the “discrete series,” the “principal series” and the “complementary series.” Discrete series representations are the square integrable ones; they can be realized on the spaces of L^2 -holomorphic sections of G_0 -homogeneous holomorphic line bundles over the open orbits. Principal series representation can be realized on L^2 (relative to invariant measure for the rotation subgroup of G_0) sections of certain G_0 -homogeneous complex line bundles over the closed orbit. The complementary series does not contribute to Plancherel measure.

In general we start with a complex manifold $X = G/P$, where G is a complex semisimple Lie group and P is a complex subgroup of the sort called *parabolic*. The latter means any of the following conditions, which are equivalent:

- $X = G/P$ is compact,
- X is a compact simply connected kaehler manifold,
- the complex manifold X is a projective variety,
- X is a closed G -orbit in a projective representation.

Then X is a *complex flag manifold*. Let G_0 be a real form of G , i.e. a real Lie group whose complexification is G . Then G_0 acts on X as a subgroup of G . It turns out that there are only a finite number of

G_0 -orbits on X and that there is just one closed orbit. The decomposition of X into G_0 -orbits, and especially the structure of the open orbits, is studied in some detail in Chapter I.

We then take an arbitrary orbit $G_0(x) \subset X$ and partition it into "maximal complex analytic pieces" that we call its *holomorphic arc components*. If $z \in G_0(x)$ then $S_{[z]}$ denotes the holomorphic arc component of $G_0(x)$ through z and $N_{[z],0}$ denotes the identity component of the Lie subgroup $\{g \in G_0: gS_{[z]} = S_{[z]}\}$; then $S_{[z]} = N_{[z],0}(z)$ so it is a real submanifold of X . The best situation is that in which some (hence every) $S_{[z]} \subset G_0(x)$ carries a positive $N_{[z],0}$ -invariant Radon measure; then we say that $G_0(x)$ is *measurable*. If $G_0(x)$ is measurable and $z \in G_0(x)$ then $S_{[z]}$ is a complex submanifold of X , in fact is an open $N_{[z],0}$ -orbit on the complex flag submanifold $N_{[z],0}^c(z)$ of X , and its invariant measure is the volume element of an $N_{[z],0}$ -invariant, possibly indefinite signature, kaehler metric. For example, the closed G_0 -orbit on X is always measurable, and we have a method for deciding measurability of the open G_0 -orbits. Holomorphic arc components and global conditions on them such as measurability of the orbit, are studied in Chapter II.

In Chapter III we work out the hermitian symmetric case in complete detail, extending earlier work of Korányi-Wolf [2], Wolf-Korányi [15] and Takeuchi [9]. This works well because every orbit is measurable. The Riemann sphere, described above, is the simplest of the hermitian symmetric cases; in fact its extreme simplicity can be misleading, so we look at the example of a general complex Grassmann manifold.

Let X be the complex Grassmann manifold consisting of the k -planes through the origin in \mathbf{C}^n . If $\{v_1, \dots, v_k\} \subset \mathbf{C}^n$ is linearly independent then $v_1 \wedge \dots \wedge v_k \in X$ denotes its span. The complex general linear group acts on X by $g: v_1 \wedge \dots \wedge v_k \rightarrow g(v_1) \wedge \dots \wedge g(v_k)$; there the scalars act trivially so the resulting group of transformations is the complex Lie group

$$G = \mathbf{GL}(n, \mathbf{C}) / \{aI: a \neq 0\} = \mathbf{SL}(n, \mathbf{C}) / \{e^{2\pi i r/n} I\}$$

of complex dimension $n^2 - 1$. Let $\{e_1, \dots, e_n\}$ be a fixed basis of \mathbf{C}^n and $x_0 = e_1 \wedge \dots \wedge e_k \in X$ our base point. Then the isotropy subgroup of G at x_0 is the complex subgroup

$$P = \{g \in \mathbf{SL}(n, \mathbf{C}): gx_0 = x_0\} / \{e^{2\pi i r/n} I\}$$

of complex dimension $k^2 + (n - k)^2 + nk - 1$. That exhibits X as a complex flag manifold² G/P of complex dimension $k(n - k)$.

Consider the hermitian form $\langle \sum_1^n z^i e_i, \sum_1^n w^j e_j \rangle = - \sum_1^k z^i \bar{w}^i + \sum_{k+1}^n z^i \bar{w}^i$ of signature $(k, n - k)$ on \mathbb{C}^n . The subgroup of $GL(n, \mathbb{C})$ preserving it is the indefinite unitary group $U^k(n)$. The latter acts on X as the real group

$$G_0 = U^k(n) / \{ e^{2\pi i \theta} I : \theta \text{ real} \} = SU^k(n) / \{ e^{2\pi i \theta / n} I \}$$

whose complexification is G . If $x \in X$ we associate a triple (a, b, c) where $\langle \cdot, \cdot \rangle$ has rank $a + b$ on x , with a negative squares and b positive ones, and $a + b + c = k$. The only restrictions are

$$a + b + c = k, \quad 0 \leq a \leq k, \quad 0 \leq b \leq n - k, \quad 0 \leq c \leq \min(k, n - k).$$

If $y \in G_0(x)$ it gives the same triple; if $y \in X$ gives the same triple then Witt's Theorem says $y \in G_0(x)$. Thus there are precisely $\frac{1}{2}(l + 1)(l + 2)$ G_0 -orbits on X , $l = \min(k, n - k)$, given by

$$G_0(c_1 c_2 \cdots c_s c_{s+1}^2 c_{s+2}^2 \cdots c_l^2 x_0), \quad 0 \leq s \leq t \leq l = \min(k, n - k),$$

where c_i are the partial Cayley transforms defined by

$$c_i(e_i) = \frac{1}{\sqrt{2}}(e_i + \sqrt{-1}e_{k+i}), \quad c_i(e_{k+i}) = \frac{1}{\sqrt{2}}(\sqrt{-1}e_i + e_{k+i}),$$

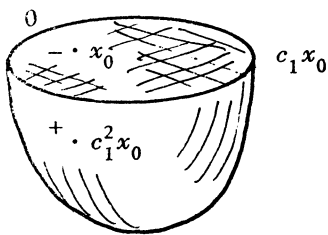
$$c_i(e_j) = e_j \quad \text{if } i \neq j \neq k + i.$$

For $(\prod_{1 \leq i \leq s} c_i)(\prod_{s < j \leq t} c_j^2)x_0$ has "signature" $(k - t, t - s, s)$. A glance at determinants shows that there are

$$l + 1 \text{ open orbits } G_0(c_1^2 c_2^2 \cdots c_l^2 x_0), \quad 0 \leq t \leq l, \\ \text{just one closed orbit } G_0(c_1 c_2 \cdots c_l x_0).$$

More precisely, the closure of $G_0(c_1 c_2 \cdots c_s c_{s+1}^2 c_{s+2}^2 \cdots c_l^2 x_0)$ is the union of all $G_0(c_1 c_2 \cdots c_u c_{u+1}^2 c_{u+2}^2 \cdots c_l^2 x_0)$ such that $v - u \leq t - s$ and $t \leq v$. Thus, for example, in the case $1 = k < n$ of complex projective $(n - 1)$ -space (of which the Riemann sphere is the case $n = 2$) there are

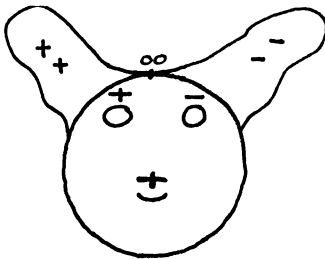
² Alternatively one can embed X in complex projective space of dimension $\binom{n}{k} - 1$ by Plücker coordinates as follows. If $x \in X$ has basis $\{v_1, \dots, v_k\}$, say $v_i = \sum_j a_{ij}^i e_j$, then x goes to the point with homogeneous coordinates given by the determinants of the $\binom{n}{k}$ matrices $(a^{i1}, a^{i2}, \dots, a^{ik}), 1 \leq j_1 < \dots < j_k \leq n$, where $a^i = (a_{i1}^i, \dots, a_{i k}^i)$. The same embedding is achieved as the closed orbit of the projective representation of $SL(n, \mathbb{C})$ associated to the k th alternation $\Lambda^k(\alpha)$ of the usual (vector) representation α of degree n .



$k = 1, n \geq 2$

- : open: negative definite lines in \mathbb{C}^n
- + : open: positive definite lines in \mathbb{C}^n
- 0 : closed: isotropic lines in \mathbb{C}^n

3 orbits, and in the case $2 = k \leq n/2$ there are 6 orbits, whose boundary structure is illustrated in the figures.³



$k = 2, n \geq 4$

- : $G_0(x_0)$ open: negative definite planes
- + - : $G_0(c_1^2 x_0)$ open: indefinite nondegenerate planes
- ++ : $G_0(c_1^2 c_2^2 x_0)$ open: positive definite planes
- +0 : $G_0(c_1 c_2^2 x_0)$ intermediate: degenerate positive semidefinite planes
- 0 : $G_0(c_1 x_0)$ intermediate: degenerate negative semidefinite planes
- 00 : $G_0(c_1 c_2 x_0)$ closed: totally isotropic planes

Back with the general complex grassmannian, we note that the open orbit $G_0(c_1^2 c_2^2 \cdots c_t^2 x_0)$ consisting of the k -planes of signature $(k-t, t)$ has coset space representation $\mathbf{U}^k(n)/\mathbf{U}^t(k) \times \mathbf{U}^{k-t}(n-k)$; so it has a G_0 -invariant indefinite-kaehler metric of signature $(t(n-2k), k(n-k) - t(n-2k))$ whose volume element will prove useful. Indefiniteness of the metric is related to the existence of compact subvarieties. G_0 has maximal compact subgroup

$$K = \{ \mathbf{U}(k) \times \mathbf{U}(n-k) \} / \{ e^{2\pi i \theta} I : \theta \text{ real} \}$$

from the maximal compact subgroup $\mathbf{U}(k) \times \mathbf{U}(n-k) \subset \mathbf{U}^k(n)$. Now

$$K(c_1^2 c_2^2 \cdots c_t^2 x_0) = \{ \mathbf{U}(k)/\mathbf{U}(t) \times \mathbf{U}(k-t) \} \times \{ \mathbf{U}(n-k)/\mathbf{U}(k-t) \times \mathbf{U}(n-2k+t) \}$$

which is the direct product of the complex grassmannians of t -planes in \mathbb{C}^k and $(k-t)$ -planes in \mathbb{C}^{n-k} . Moreover the maximal compact sub-

³ The figure for $k=2$ is due to A. Korányi.

varieties of the open orbits are the

$$g \cdot K(c_1^2 c_2^2 \cdots c_t^2 x_0) \subset G_0(c_1^2 c_2^2 \cdots c_t^2 x_0), \quad g \in G_0.$$

Furthermore there exist holomorphic fibrations

$$\beta_t: G_0(c_1^2 c_2^2 \cdots c_t^2 x_0) \rightarrow K(c_1^2 c_2^2 \cdots c_t^2 x_0)$$

given as follows. First notice that the intended base space of β_t consists of all $v_1 \wedge \cdots \wedge v_{k-t} \wedge w_1 \wedge \cdots \wedge w_t$ where $v_1 \wedge \cdots \wedge v_{k-t}$ is a subspace of $e_1 \wedge \cdots \wedge e_k$ and $w_1 \wedge \cdots \wedge w_t$ is a subspace of $e_{k+1} \wedge \cdots \wedge e_n$. If $x \in X$ has signature $(k-t, t)$, its orthogonal projection to $e_1 \wedge \cdots \wedge e_k$ is a subspace $v_1 \wedge \cdots \wedge v_{k-t}$, and its orthogonal projection to $e_{k+1} \wedge \cdots \wedge e_n$ is some $w_1 \wedge \cdots \wedge w_t$; by definition $\beta_t(x) = v_1 \wedge \cdots \wedge v_{k-t} \wedge w_1 \wedge \cdots \wedge w_t$. It is not too hard to check that this defines a holomorphic fibre bundle with total space and base as mentioned, whose fibre over $k(c_1^2 c_2^2 \cdots c_t^2 x_0)$, $k \in K$, is naturally isomorphic to the product of open orbits in smaller grassmannians, negative definite t -planes in a \mathbf{C}^{2t} of signature (t, t) and

negative definite $(k-t)$ -planes in a \mathbf{C}^{n-2t} of signature $(k-t, n-k-t)$.

One can use the maximal compact subvarieties to check directly that the complex manifold $G_0(c_1^2 c_2^2 \cdots c_t^2 x_0)$ has nonconstant holomorphic functions if, and only if, either $t=0$ or $t=k=n/2$. Thus the closest "approximation" to holomorphic functions on $G_0(c_1^2 c_2^2 \cdots c_t^2 x_0)$ is that of holomorphic sections of holomorphic line bundles $\beta_t^* \mathcal{L}$ where $\mathcal{L} \rightarrow K(c_1^2 c_2^2 \cdots c_t^2 x_0)$ is a holomorphic line bundle with nonzero positive semidefinite Chern class. Look back to the case $k=1$ of complex projective $(n-1)$ -space. There $G_0(x_0)$ is the unit ball in \mathbf{C}^{n-1} , which carries holomorphic functions, and β_0 is fibration over the single point x_0 . The other open orbit $G_0(c_1^2 x_0)$ is the exterior of that unit ball with a complex projective $(n-2)$ -space $K(c_1^2 x_0)$ attached; if $n > 2$ it carries no nonconstant holomorphic functions; β_1 projects it to the polar hyperplane $K(c_1^2 x_0)$ of x_0 , and eligible bundles $\beta_1^* \mathcal{L}$ are the $\beta_1^* \mathcal{H}^q$, $q \geq 1$, where $\mathcal{H} \rightarrow K(c_1^2 x_0)$ is the hyperplane bundle over that projective space.

We now cut an arbitrary G_0 -orbit on X into complex submanifolds of X . If $g \in G$ (resp. $g \in G_0$) then $\tilde{g} \in \mathbf{SL}(n, \mathbf{C})$ (resp. $\tilde{g} \in \mathbf{SU}^k(n)$) denotes a representing linear transformation of \mathbf{C}^n . Now define, for $0 \leq s \leq l = \min(k, n-k)$,

$$G_{[s]} = \{g \in G: \tilde{g}e_i = e_i \text{ and } \tilde{g}e_{k+i} = e_{k+i} \text{ for } 1 \leq i \leq s\},$$

$$G_{[s],0} = \{g \in G_0: \tilde{g}e_i = e_i \text{ and } \tilde{g}e_{k+i} = e_{k+i} \text{ for } 1 \leq i \leq s\}.$$

Then

$$G_{[s]}(c_1c_2 \cdots c_sx_0) = G_{[s]}(c_1c_2 \cdots c_s c_{s+1}^2 c_{s+2}^2 \cdots c_t^2 x_0)$$

for $s \leqq t \leqq l$, and that manifold is the sub-grassmannian of X consisting of all $(k-s)$ -planes in the \mathbb{C}^{n-2s} with basis $\{e_{s+1}, \cdots, e_k; e_{k+s+1}, \cdots, e_n\}$. Furthermore

$$G_{[s],0}(c_1c_2 \cdots c_s c_{s+1}^2 c_{s+2}^2 \cdots c_t^2 x_0) \text{ is open in } G_{[s]}(c_1c_2 \cdots c_sx_0),$$

and thus is a complex submanifold of X with an invariant indefinite-kaehler metric of signature

$$((t-s)(n-2s-2k), \quad k(n-2s-k) - (t-s)(n-2s-2k)).$$

Now the point is that $G_0(c_1c_2 \cdots c_s c_{s+1}^2 c_{s+2}^2 \cdots c_t^2 x_0)$ is a union of submanifolds $k \cdot G_{[s],0}(c_1c_2 \cdots c_s c_{s+1}^2 c_{s+2}^2 \cdots c_t^2 x_0)$ with $k \in K$, that any two such submanifolds are equal or disjoint, and that any holomorphic map of the unit disc into X with image in

$$G_0(c_1c_2 \cdots c_s c_{s+1}^2 c_{s+2}^2 \cdots c_t^2 x_0)$$

must have image in one of those submanifolds. Thus every G_0 -orbit is decomposed into a disjoint union of indefinite-kaehler submanifolds of X that are maximal among complex submanifolds of X contained in the orbit. They are the *holomorphic arc components* of the orbit. It now follows that every G_0 -orbit on X is *measurable*.

Let $y = c_1c_2 \cdots c_s c_{s+1}^2 c_{s+2}^2 \cdots c_t^2 x_0$. Now every holomorphic arc component of the arbitrary G_0 -orbit $G_0(y) \subset X$ is an open orbit $kG_{[s],0}k^{-1}(ky)$, $k \in K$, on the smaller grassmannian $kG_{[s]}(y)$. Thus it has a holomorphic fibration

$$\beta_{(k)}: kG_{[s],0}(y) \rightarrow k(K \cap G_{[s],0})(y)$$

over its maximal compact subvariety. It turns out that the $\beta_{(k)}$ fit together in the sense that

$$\text{if } k_1G_{[s],0}(y) = k_2G_{[s],0}(y) \text{ then } \beta_{(k_1)} = \beta_{(k_2)}.$$

The result is a K -equivariant fibration

$$\beta: G_0(y) \rightarrow K(y)$$

whose fibres coincide with those of its restrictions $\beta_{(k)}$, and which thus projects every holomorphic arc component holomorphically to its maximal compact subvariety. Thus one obtains "generalized Siegel domain" realizations of $G_0(y)$ by fitting together the standard Siegel domain realizations of the β -fibres.

Finally, let me describe the expected relation to representation theory. Let $G_0(x) \subset X$ be a measurable orbit. Consider G_0 -homogeneous complex vector bundles $\mathcal{E} \rightarrow G_0(x)$ that are holomorphic over every holomorphic arc component of $G_0(x)$. Given \mathcal{E} we form the cohomology groups $H^p(\mathcal{E})$ consisting of the \mathcal{E} -valued p -forms ω on $G_0(x)$ such that ω is harmonic and of type $(0, p)$ on every holomorphic arc component and

$$\int_K \int_{k \cdot S_{[x]}} \|\omega\|^2 d\mu_k dk < \infty.$$

Here $d\mu_k$ is the (normalized suitably) invariant Radon measure on the arbitrary holomorphic arc component $k \cdot S_{[x]}$, and dk is Haar measure on a maximal compact subgroup K of G_0 . Now the point is to choose \mathcal{E} so that the action of G_0 on $H^p(\mathcal{E})$ is unitary. The evidence is good that measurable open orbits give the discrete series of G_0 , and the (necessarily measurable) closed orbit gives the principal series. The process can be viewed as a specialization of Kostant's theory of symplectic manifolds and polarizations ([4]; see [5] for details) that is set up for semisimple groups. I will go into it in fair detail in [12].

The first third (Chapter I) of this paper represents part of some joint work I did with Bertram Kostant, and that work influenced the second third (Chapter II). Also, a result of Masaru Takeuchi [9] is included in Chapter III for expository reasons. Thus only about 4/5 of this paper is my own work. For that reason there are chapter notes distributing credit in detail at the ends of the introductions to each of the three chapters.

CHAPTER I. DECOMPOSITION OF A COMPLEX FLAG MANIFOLD INTO REAL GROUP ORBITS

We establish the basic notions for G_0 -orbits on a complex flag manifold $X = G/P$, G_0 a real form of G , in §2; there the root decompositions are set up, the number of G_0 -orbits is bounded, open orbits are shown to exist, and dimensions and codimensions of orbits are computed in such a manner that open orbits are easy to manage. In §3 it is shown that there is a unique closed G_0 -orbit on X , and various properties are explored for that closed orbit. §4 is the initial assault on open orbits; the structure of open orbits is worked out in detail sufficient to supply an enumeration procedure. Open orbits are studied as homogeneous spaces in §5, where their maximal compact subvarieties are described, where they are proved to be simply connected, and where their holomorphic functions are explicitly described. In §6 we study the existence problem for invariant Radon measures on open orbits and

find a complete solution. Finally, in §7, the notion of integrable orbit (which is important for Chapter II and application to group representations) is worked out in detail sufficient for an enumeration.

NOTES FOR CHAPTER I. The results of §2, §3 and §4 were proved in some conversations between Bertram Kostant and this author. Theorem 2.6 is essentially the work of Kostant while Theorem 2.12 is due to this author. The material surrounding Theorem 3.3 is primarily the work of Kostant; Theorem 3.6 and its corollaries are essentially due to this author. Theorem 4.5 is due more to Kostant than to this author, and Theorem 4.9 slightly more to this author than to Kostant.

The results of §5 were worked out by this author. Lemma 5.1 and Theorem 5.4 are easy in the case where $G_0 \cap P_z$ is compact and our proof may be viewed as a reduction to that case. Theorem 5.7 had been suspected by P. A. Griffiths in the case where $G_0 \cap P_z$ is compact and G is simple; he felt that one should be able to join points of $G_0(x)$ by a chain of compact subvarieties under appropriate circumstances; our proof uses a Lie algebra maximality criterion instead.

Theorem 6.3 and its corollaries were worked out jointly in conversations between the author and Bertram Kostant. Then Kostant suggested an examination of the case where G_0 is complex; that case is described at the end of §6; it led to this author working out Theorem 6.7 and its corollary.

The notion of integrable orbit was suggested by Bertram Kostant's work on symplectic homogeneous spaces. Equivalence of (i), (ii) and (iii) in Theorem 7.2 was worked out jointly in conversations between the author and Kostant, and perhaps is due mostly to Kostant. The rest of §7 is due to this author.

The conversations between Bertram Kostant and the author that resulted in material in Chapter I took place between August 1965 and February 1967.

2. Basic facts on the orbit structure. Let G be a complex semisimple connected Lie group. \mathfrak{g} is its Lie algebra. Choose a Cartan subalgebra $\mathfrak{H} \subset \mathfrak{g}$ and a system Π of simple roots. If φ is a root we denote its coefficients by

$$\varphi = \sum_{\pi \in \Pi} n_{\varphi}(\pi) \cdot \pi.$$

Suppose $\Phi \subset \Pi$. Then the root system Δ is a disjoint union,

$$(2.1a) \quad \Delta = \Phi^r \cup \Phi^u \cup -\Phi^u = \Phi^* \cup -\Phi^u$$

where we define $\Phi^* = \Phi^r \cup \Phi^u$ and

$$(2.1b) \quad \Phi^r = \{ \varphi \in \Delta : n_\varphi(\pi) = 0 \text{ for all } \pi \in \Pi - \Phi \},$$

$$(2.1c) \quad \Phi^u = \{ \varphi \in \Delta : n_\varphi(\pi) > 0 \text{ for some } \pi \in \Pi - \Phi \}.$$

That defines subalgebras

$$(2.2) \quad \mathcal{O}_\Phi^r = \mathcal{K} + \sum_{\varphi^r} \mathcal{G}_\varphi, \quad \mathcal{O}_\Phi^u = \sum_{\varphi^u} \mathcal{G}_\varphi, \quad \mathcal{O}_\Phi = \mathcal{O}_\Phi^r + \mathcal{O}_\Phi^u$$

of \mathcal{G} . \mathcal{O}_Φ^r is reductive, \mathcal{O}_Φ^u is unipotent, \mathcal{O}_Φ is their semidirect sum; \mathcal{O}_Φ is the normalizer of \mathcal{O}_Φ^u in \mathcal{G} . $\mathcal{K}_\Phi = \{ h \in \mathcal{K} : \Phi(h) = 0 \}$ is the center of \mathcal{O}_Φ^r .

2.3. DEFINITION. \mathcal{O}_Φ is the **parabolic subalgebra** of \mathcal{G} defined by \mathcal{K} , Π and Φ . If Φ is empty, so $\mathcal{O}_\Phi = \mathcal{K} + \sum_{\varphi > 0} \mathcal{G}_\varphi$, then \mathcal{O}_Φ is the **Borel subalgebra** of \mathcal{G} defined by \mathcal{K} and Π .

It is standard that

(i) the Borel subalgebras of \mathcal{G} are just the maximal solvable subalgebras,

(ii) the parabolic subalgebras of \mathcal{G} are just the subalgebras which contain Borel subalgebras of \mathcal{G} , and

(iii) given \mathcal{K} , Π and a conjugacy class (inner automorphisms) $\{ \mathcal{O} \}$ of parabolic subalgebras of \mathcal{G} , there is a unique subset $\Phi \subset \Pi$ such that $\mathcal{O}_\Phi \in \{ \mathcal{O} \}$.

2.4. DEFINITION. A **Borel subgroup** of G is a maximal complex solvable Lie subgroup. A **parabolic subgroup** of G is a complex Lie subgroup which contains a Borel subgroup.

It is standard that the following conditions are equivalent for a complex Lie subgroup $L \subset G$:

(i) L is a parabolic subgroup,

(ii) the Lie algebra of L is a parabolic subalgebra of \mathcal{G} ,

(iii) L is the analytic subgroup of G for a parabolic subalgebra of \mathcal{G} , and

(iv) G/L is compact.

The Borel subgroups are the parabolic subgroups whose Lie algebras are Borel subalgebras of \mathcal{G} . A parabolic subgroup is connected and equal to its own normalizer in G .

Let $P \subset G$ be a parabolic subgroup with Lie algebra \mathcal{O} . Then \mathcal{G} has a Cartan subalgebra $\mathcal{K} \subset \mathcal{O}$, and there is a system Π of simple roots and a subset $\Phi \subset \Pi$, such that $\mathcal{O} = \mathcal{O}_\Phi$. Let P^r and P^u denote the respective analytic subgroups of P for $\mathcal{O}^r = \mathcal{O}_\Phi^r$ and $\mathcal{O}^u = \mathcal{O}_\Phi^u$. Then $P^r \cap P^u = \{ 1 \}$, so $P = P^r \cdot P^u$ semidirect product, and P is the normalizer of P^u in G .

2.5. DEFINITION. A **complex flag manifold** is a coset space $X = G/P$, where G is a complex semisimple connected Lie group and P is a parabolic subgroup; X is viewed as a complex manifold and G is viewed as a group of holomorphic transformations of X .

Let $X = G/P$ be a complex flag manifold. If $x \in X$, say $x = gP$, then $\text{ad}(g)P = gPg^{-1}$ is the isotropy subgroup of G at x . As P is its own normalizer in G we may identify x with $\text{ad}(g)P$, viewing X as the space of parabolic subgroups of G conjugate to P . Whether we do that or not, we often write P_x for the parabolic $\text{ad}(g)P$, $x = gP$, corresponding to x .

Let G_0 be a real form of G . In other words, G_0 is the analytic subgroup for a real form \mathfrak{g}_0 of \mathfrak{g} . τ denotes complex conjugation of \mathfrak{g} over \mathfrak{g}_0 , of G over G_0 ; so G_0 is the identity component of the fixed point set of τ on G .

2.6. THEOREM. *Let $X = G/P$ be a complex flag manifold. Let G_0 be a real form of G and let τ be conjugation of G over G_0 .*

1. *If $x \in X$ then the isotropy subalgebra $\mathfrak{g}_0 \cap \mathcal{O}_x$ is a real form of $\mathcal{O}_x \cap \tau\mathcal{O}_x$ and contains a Cartan subalgebra of \mathfrak{g}_0 . Conversely if \mathfrak{K}_0 is a Cartan subalgebra of \mathfrak{g}_0 then $\mathfrak{K}_0 \subset \mathfrak{g}_0 \cap \mathcal{O}_x$ for some $x \in X$.*

2. *Let m be the number of conjugacy classes of Cartan subalgebras of \mathfrak{g}_0 . Let W_G denote the Weyl group of G . Then G_0 has $\leq m|W_G|$ orbits on X .*

3. *G_0 has an open orbit on X , and the union of the open G_0 -orbits is dense in X .*

PROOF. $\mathfrak{g}_0 \cap \mathcal{O}_x$ is the isotropy subalgebra of \mathfrak{g}_0 at x , and $\mathfrak{g}_0 \cap \mathcal{O}_x = (\mathfrak{g}_0 \cap \mathcal{O}_x) \cap \tau(\mathfrak{g}_0 \cap \mathcal{O}_x) = \mathfrak{g}_0 \cap \mathcal{O}_x \cap \tau\mathcal{O}_x$; so $\mathfrak{g}_0 \cap \mathcal{O}_x$ is a real form of $\mathcal{O}_x \cap \tau\mathcal{O}_x$.

Choose a Borel subalgebra $\mathfrak{B} \subset \mathcal{O}_x$ of \mathfrak{g} . Then $\mathfrak{B} \cap \tau\mathfrak{B}$ is τ -stable and contains a Cartan subalgebra of \mathfrak{g} , so it contains a τ -stable Cartan subalgebra \mathfrak{K} of \mathfrak{g} . Now $\mathfrak{K} \cap \mathfrak{g}_0$ is a Cartan subalgebra of \mathfrak{g}_0 contained in $\mathfrak{g}_0 \cap \mathcal{O}_x$.

Let \mathfrak{K}_0 be a Cartan subalgebra of \mathfrak{g}_0 and let $\mathfrak{K} = \mathfrak{K}_0^c$, Cartan subalgebra of \mathfrak{g} . Choose a system Π of simple roots and a subset $\Phi \subset \Pi$ such that \mathcal{O}_Φ is conjugate to \mathcal{O} in \mathfrak{g} . Then $\mathcal{O}_\Phi = \mathcal{O}_x$ for some $x \in X$, and $\mathfrak{K}_0 \subset \mathfrak{g}_0 \cap \mathcal{O}_x$.

This completes the proof of (1).

Let $\{\mathfrak{K}_1, \dots, \mathfrak{K}_m\}$ represent the conjugacy classes of Cartan subalgebras of \mathfrak{g}_0 . Given \mathfrak{K}_i we have $|W_G|$ systems Π of simple \mathfrak{K}_i^c -roots of \mathfrak{g} . Given Π we have a unique subset $\Phi \subset \Pi$ such that $\mathcal{O}_\Phi = \mathcal{O}_x$ for some $x \in X$, and that specifies an orbit $G_0(x)$. We have just described $m \cdot |W_G|$ orbits of G_0 on X ; they need not be distinct, but we must check that every G_0 -orbit is one of them. Let $z \in X$. Then \mathfrak{g}_0 has a Cartan subalgebra $\mathfrak{K}_0 \subset \mathfrak{g}_0 \cap \mathcal{O}_z$. We have $1 \leq i \leq m$ and $g \in G$ such that $\text{ad}(g)\mathfrak{K}_0 = \mathfrak{K}_i$; now $\mathfrak{K}_i \subset \mathfrak{g}_0 \cap \mathcal{O}_{g(z)}$ so we have a Π and a subset $\Phi \subset \Pi$ such that $\mathcal{O}_{g(z)} = \mathcal{O}_\Phi$; thus $G_0(z) = G_0(x)$ where $\mathcal{O}_\Phi = \mathcal{O}_x$. This proves (2).

Now G_0 has only a finite number of orbits on X . Thus some G_0 -orbit has full dimension; hence is open, and the union of the lower dimensional G_0 -orbits is a closed lower dimensional subset of X . This proves (3). Q.E.D.

Now the case of compact G_0 , which is not new, becomes rather easy:

2.7. COROLLARY. *Let $X = G/P$ be a complex flag manifold. Let G_u be a compact real form of G . Then G_u is transitive on X , so $X = G_u/(G_u \cap P)$, and the isotropy subgroup $G_u \cap P$ is the centralizer of a torus. In particular X is simply connected, the integral cohomology $H^*(X; \mathbf{Z})$ is torsion free, and $H^q(X; \mathbf{Z}) = 0$ for q odd.*

PROOF. Let $G_u(x)$ be an open orbit. It is closed because G_u is compact. Thus $G_u(x) = X$.

Let $\mathfrak{K}_u \subset \mathfrak{G}_u \cap \mathfrak{P}$ be a Cartan subalgebra of \mathfrak{G}_u , $\mathfrak{K} = \mathfrak{K}_u^{\mathbb{C}}$, Π a simple root system, and $\Phi \subset \Pi$, such that $\mathfrak{P} = \mathfrak{P}_{\Phi}$. $i\mathfrak{K}_u$ contains the roots so $\tau\varphi = -\varphi$ for every root φ . Thus $\tau\mathfrak{P} \cap \mathfrak{P} = \mathfrak{P}^{\sigma}$, centralizer in \mathfrak{G} of $\mathfrak{K}_{\Phi} = \{h \in \mathfrak{K} : \Phi(h) = 0\}$; now $\mathfrak{P}_u^{\sigma} = \mathfrak{G}_u \cap \mathfrak{P}$ is a real form of \mathfrak{P}^{σ} , centralizer in \mathfrak{G}_u of $\mathfrak{s} = \{h \in \mathfrak{K}_u : \Phi(h) = 0\}$. Let S be the toral subgroup of G_u with Lie algebra \mathfrak{s} . Now the centralizer of S in G_u is connected and contains $G_u \cap P$, and is the identity component of $G_u \cap P$. Thus $G_u \cap P$ is the centralizer of S ; in particular X is simply connected and it follows from results of Bott-Samelson [1] that $H^*(X, \mathbf{Z})$ is torsion free, nonzero only in even degrees. Q.E.D.

2.8. COROLLARY. *Let $P_1 \subset P_2$ be parabolic subgroups of G so $X_i = G/P_i$ are complex flag manifolds. Let $\pi : X_1 \rightarrow X_2$ be the natural G -equivariant fibration. Then the fibre of π is a complex flag manifold $Y = (P_2')/P_1 \cap (P_2)'$, and, relative to any coefficient ring, Poincaré polynomials satisfy*

$$p(X_1, t) = p(X_2, t) \cdot p(Y, t).$$

PROOF. π has fibre $F = P_2/P_1$. Choose a Cartan subalgebra $\mathfrak{K} \subset \mathfrak{P}_1$ of \mathfrak{G} , a system Π of simple roots, and subsets $\Phi_1 \subset \Phi_2$ of Π , such that $\mathfrak{P}_i = \mathfrak{P}_{\Phi_i}$. Then $\mathfrak{P}_1^{\sigma} \subset \mathfrak{P}_2^{\sigma}$ and $\mathfrak{P}_1^{\sigma} \supset \mathfrak{P}_2^{\sigma}$, so P_2^{σ} acts trivially and P_1^{σ} acts transitively on F . As \mathfrak{P}_1 contains the center of \mathfrak{P}_2^{σ} , the semisimple part $(P_2^{\sigma})'$ acts transitively on F , and its isotropy subgroup $P_1 \cap (P_2^{\sigma})'$ is a parabolic subgroup. This realizes F as the complex flag manifold Y . All three of X_1 , X_2 and Y now have no integral cohomology in odd degrees and no torsion in even degrees, so the spectral sequence of π is trivial. Q.E.D.

We take a closer look at the G_0 -orbits on $X = G/P$. Let $x \in X$, choose a Cartan subalgebra $\mathfrak{K}_0 \subset \mathfrak{G}_0 \cap \mathfrak{P}_x$ of \mathfrak{G}_0 , let $\mathfrak{K} = \mathfrak{K}_0^{\mathbb{C}}$, and choose

a simple root system Π and a subset Φ such that $\mathcal{O}_x = \mathcal{O}_\Phi$. Now we define

(2.9a) $l = \text{rank } \mathfrak{G}, \text{ i.e. } l = \dim_C \mathfrak{K} = \dim_{\mathbb{R}} \mathfrak{K}_0,$

(2.9b) $r = \text{number of positive roots of } \mathfrak{G},$

(2.9c) $r_\Phi = \text{number of positive roots of } \mathcal{O}_\Phi^r.$

Then $\dim_C G = l + 2r, \dim_C \mathcal{O}^r = l + 2r_\Phi, \dim_C \mathcal{O}^u = r - r_\Phi, \dim_C \mathcal{O} = l + r + r_\Phi$ and $\dim_C X = r - r_\Phi$.

2.10. LEMMA. $\mathcal{O}_x \cap \tau \mathcal{O}_x$ is the semidirect sum of the reductive algebra $(\mathcal{O}_x \cap \tau \mathcal{O}_x)^r$ given by

(2.11a)
$$\mathcal{O}_\Phi^r \cap \tau \mathcal{O}_\Phi^r = \mathfrak{K} + \sum_{\Phi^r \cap \tau \Phi^r} \mathfrak{G}_\varphi$$

with the nilpotent ideal $(\mathcal{O}_x \cap \tau \mathcal{O}_x)^u$ given by

(2.11b)
$$\begin{aligned} & (\mathcal{O}_\Phi^r \cap \tau \mathcal{O}_\Phi^u) + (\mathcal{O}_\Phi^u \cap \tau \mathcal{O}_\Phi^r) + (\mathcal{O}_\Phi^u \cap \tau \mathcal{O}_\Phi^u) \\ &= \left\{ \sum_{\Phi^r \cap \tau \Phi^u} + \sum_{\Phi^u \cap \tau \Phi^r} + \sum_{\Phi^u \cap \tau \Phi^u} \right\} (\mathfrak{G}_\varphi). \end{aligned}$$

In particular

$$\dim_{\mathbb{R}}(\mathfrak{G}_0 \cap \mathcal{O}_x) = \dim_C(\mathcal{O}_x \cap \tau \mathcal{O}_x) = \dim_C \mathcal{O}_\Phi^r + |\Phi^u \cap \tau \Phi^u|.$$

PROOF. $\mathcal{O}_x \cap \tau \mathcal{O}_x$ is the sum of \mathfrak{K} and certain \mathfrak{G}_φ , with \mathfrak{G}_φ in the reductive part (resp. nilpotent ideal) if and only if $\mathfrak{G}_{-\varphi}$ also occurs (resp. does not occur) in $\mathcal{O}_x \cap \tau \mathcal{O}_x$. Thus $(\mathcal{O}_x \cap \tau \mathcal{O}_x)^r$ is the sum of \mathfrak{K} and $\{\mathfrak{G}_\varphi: \text{both } \varphi \text{ and } -\varphi \text{ are in } \Phi^* \cap \tau \Phi^*\}$, i.e. the sum of \mathfrak{K} and $\{\mathfrak{G}_\varphi: \varphi \in \Phi^r \cap \tau \Phi^r\}$, as asserted, and $(\mathcal{O}_x \cap \tau \mathcal{O}_x)^u$ is the sum of all \mathfrak{G}_φ for $\varphi \in \Phi^* \cap \tau \Phi^*$ but $\varphi \notin \Phi^r \cap \tau \Phi^r$, i.e. $\varphi \in (\Phi^r \cap \tau \Phi^u) \cup (\Phi^u \cap \tau \Phi^r) \cup (\Phi^u \cap \tau \Phi^u)$ as asserted.

Now $\dim_C(\mathcal{O}_x \cap \tau \mathcal{O}_x) = l + |\Phi^r \cap \tau \Phi^r| + |\Phi^r \cap \tau \Phi^u| + |\Phi^u \cap \tau \Phi^r| + |\Phi^u \cap \tau \Phi^u|$. But $\dim_C \mathcal{O}_\Phi^r = l + |\Phi^r| = l + |\Phi^r \cap \tau \Phi^r| + |\Phi^r \cap \tau \Phi^u| + |\Phi^r \cap -\tau \Phi^u|$, and $\Phi^u \cap \tau \Phi^r = -\tau(\Phi^r \cap -\tau \Phi^u)$ implies $|\Phi^r \cap -\tau \Phi^u| = |\Phi^u \cap \tau \Phi^r|$. Thus $\dim_C(\mathcal{O}_x \cap \tau \mathcal{O}_x) - \dim_C \mathcal{O}_\Phi^r = |\Phi^u \cap \tau \Phi^u|$. Q.E.D.

2.12. THEOREM. Let $X = G/P$ be a complex flag manifold, $x \in X$ and $P_x = P_\Phi$ as above. Then

- (i) $\dim_C X = r - r_\Phi,$
- (ii) $|\Phi^u \cap \tau \Phi^u|$ is the real codimension of the orbit $G_0(x)$ in X , and
- (iii) $G_0(x)$ is open in X if and only if $\Phi^u \cap \tau \Phi^u$ is empty.

PROOF. (i) was noted as an immediate consequence of the definitions of r and r_Φ .

For (ii) we calculate

$$\begin{aligned} \text{codim}_{\mathbb{R}} G_0(x) &= \dim_{\mathbb{R}} X - \dim_{\mathbb{R}} G_0(x) \\ &= 2 \dim_{\mathbb{C}} X - \{ \dim_{\mathbb{R}} G_0 - \dim_{\mathbb{R}} G_0 \cap P_x \} \\ &= (2r - 2r_\Phi) - \{ (l + 2r) - (l + 2r_\Phi + | \Phi^u \cap \tau\Phi^u |) \} \\ &= | \Phi^u \cap \tau\Phi^u |. \end{aligned}$$

For (ii) we note that $G_0(x)$ is open in X if and only if $\text{codim}_{\mathbb{R}} G_0(x) = 0$. Q.E.D.

Now the orbit $G_0(x)$ has real dimension $2(r - r_\Phi) - | \Phi^u \cap \tau\Phi^u |$, maximized as $2(r - r_\Phi)$ for the open orbits, minimized for a closed orbit.

3. The closed orbit. Let $x \in X$. Then either $G_0(x)$ is closed in X or its topological boundary is a nonempty union of lower dimensional orbits. Thus any G_0 -orbit of minimal dimension is closed.

Let $G_0(x)$ be a closed orbit. In other words $G_0/(G_0 \cap P_x)$ is compact. Thus there is a maximal compact subgroup $K \subset G_0$ and an Iwasawa decomposition $G_0 = KAN$, such that AN is contained in the isotropy subgroup $(P_x \cap \tau P_x)_0 = G_0 \cap P_x$ of G_0 at x . Extend A to a Cartan subgroup $H_0 = T \times A$ of G_0 , where $T \subset K$ is a maximal torus of the centralizer M of A in K . Then MAN is a minimal parabolic subgroup of G_0 and $(P_x \cap \tau P_x)_0$ contains TAN . We have proved

3.1. LEMMA. *$G_0(x)$ is a closed G_0 -orbit in X if and only if there is an Iwasawa decomposition $G_0 = KAN$, the corresponding minimal parabolic subgroup MAN of G_0 , and a maximal torus T of M , such that $TAN \subset G_0 \cap P_x$.*

Now suppose that P is a Borel subgroup of G_0 , so $\mathcal{P}_x \cap \tau\mathcal{P}_x$ is solvable. Let $\mathcal{H} = \mathcal{H}_0^e$ and let φ be an \mathcal{H} -root. First suppose $\varphi(\mathfrak{A}) \neq 0$. If $\mathfrak{G}_\varphi \not\subset \mathfrak{H}^e$ then $\mathfrak{G}_{-\varphi} \subset \mathfrak{H}^e$, so $\mathfrak{G}_\varphi \not\subset \mathcal{P}_x \cap \tau\mathcal{P}_x$ because the latter is solvable. Now suppose $\varphi(\mathfrak{A}) = 0$ so $\mathfrak{G}_\varphi \subset \mathfrak{H}^e$. If $\mathfrak{G}_\varphi \subset \mathcal{P}_x \cap \tau\mathcal{P}_x$ then $\mathfrak{G}_{-\varphi} \subset \mathcal{P}_x \cap \tau\mathcal{P}_x$ because M is compact, contradicting solvability of $\mathcal{P}_x \cap \tau\mathcal{P}_x$. We have proved

3.2. LEMMA. *Let $X = G/B$ where B is a Borel subgroup. Let $G_0(x)$ be a closed orbit. Then there is an Iwasawa decomposition $G_0 = KAN$, the corresponding minimal parabolic subgroup MAN of G_0 , and a maximal torus T of M , such that $\mathfrak{G}_0 \cap \mathcal{P}_x = \mathfrak{J} + \mathfrak{A} + \mathfrak{H}$.*

Now we can prove uniqueness.

3.3. THEOREM. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G . Then there is a unique closed G_0 -orbit on X .*

PROOF. Let $B \subset P$ be a Borel subgroup of G and let $Y = G/B$ be the corresponding complex flag manifold. We have the natural projection $\pi: Y \rightarrow X$ with compact fibres. Let $x \in X$. Then $\pi^{-1}G_0(x)$ is a union of G_0 -orbits on Y . If $G_0(x)$ is closed, hence compact, then $\pi^{-1}G_0(x)$ is closed, so $\pi^{-1}G_0(x)$ contains a closed G_0 -orbit in Y . If $G_0(x_1)$ and $G_0(x_2)$ are distinct closed G_0 -orbits on X , now their respective inverse images contain distinct closed G_0 -orbits on Y . Thus we need only prove our uniqueness assertion on Y . In other words the proof is reduced to the case where P is a Borel subgroup of G .

Now P is a Borel subgroup of G . Fix an Iwasawa decomposition $G_0 = KAN$, the corresponding minimal parabolic subgroup MAN of G_0 , and a maximal torus T of M . If $G_0(x)$ is closed then we have other choices $G_0 = K'A'N'$, $M'A'N'$ and $T' \subset M'$, such that $\mathfrak{g}_0 \cap \mathfrak{p}_x = \mathfrak{J}' + \mathfrak{A}' + \mathfrak{N}'$. Any two Iwasawa decompositions of G_0 are conjugate. So we have $g \in G_0$ such that $\text{ad}(g)$ sends K' to K , A' to A and N' to N . Automatically $\text{ad}(g)M' = M$. Any two maximal tori of M are conjugate, and M normalizes both A and N ; thus we may also assume $\text{ad}(g)T' = T$. Replacing x by $g(x)$, the orbit $G_0(x) = G_0(gx)$ is not changed, and we may assume $\mathfrak{g}_0 \cap \mathfrak{p}_x = \mathfrak{J} + \mathfrak{A} + \mathfrak{N}$. Now the Borel subalgebra $\mathfrak{p}_x = \mathfrak{L} + \mathfrak{A}^e + \mathfrak{N}^e$ where, for some ordering, \mathfrak{L} is the sum of \mathfrak{J}^e with the positive \mathfrak{J}^e -root spaces of \mathfrak{M}^e . Conjugating further by an element $g \in M \subset G_0$ we may assume that ordering to be a given one.

Let $G_0(x_1)$ and $G_0(x_2)$ be closed orbits on X . We have just seen that x_i can be replaced within $G_0(x_i)$ so that $\mathfrak{p}_{x_i} = \mathfrak{L} + \mathfrak{A}^e + \mathfrak{N}^e$ in the notation above, i.e. so that $\mathfrak{p}_{x_1} = \mathfrak{p}_{x_2}$, i.e. so that $x_1 = x_2$. Thus $G_0(x_1) = G_0(x_2)$. Q.E.D.

An immediate consequence is

3.4. COROLLARY. *Let $X = G/P$ be a complex flag manifold, $x \in X$, and G_0 a real form of G . Then the following conditions are equivalent:*

- (i) $G_0(x)$ is the unique closed G_0 -orbit.
- (ii) $G_0(x)$ is contained in the closure of every G_0 -orbit on X .
- (iii) $G_0(x)$ is contained in every G_0 -stable closed subset of X .
- (iv) Some maximal compact subgroup of G_0 is transitive on $G_0(x)$.
- (v) Every maximal compact subgroup of G_0 is transitive on $G_0(x)$.
- (vi) $G_0(x)$ is the lowest-dimensional G_0 -orbit on X .

PROOF. Equivalence of (i), (ii), (iii) and (vi) follows from uniqueness of the closed orbit and the fact that it is on the boundary of every nonclosed orbit. Trivially (v) implies (iv) and (iv) implies (i), so we need only check that (i) implies (v). Choose a Borel subgroup $B \subset P$

of G , let $Y = G/B$ and let $\pi: Y \rightarrow X$ be the projection. $G_0(x)$ is closed, so $\pi^{-1}G_0(x)$ is closed, so $\pi^{-1}G_0(x)$ contains a closed orbit $G_0(y)$. In the notation of the lemmas, $\mathfrak{B}_y \cap \mathfrak{G}_0 = \mathfrak{J} + \mathfrak{Q} + \mathfrak{N}$ where $G_0 = KAN$. Let K' be any maximal compact subgroup of G_0 . Then $K' \cap AN = \{1\}$, so

$$\begin{aligned} \dim G_0(y) &\geq \dim K'(y) = \dim K' - \dim(K' \cap TAN) \\ &\geq \dim K' - \dim T = \dim K - \dim T \\ &= \dim K(y) = \dim K/T = \dim G_0/TAN \\ &= \dim G_0(y), \end{aligned}$$

proving $K'(y) = G_0(y)$. Now $G_0(x) = \pi G_0(y) = \pi K'(y) = K'(x)$. Q.E.D.

We just made implicit use of the obvious consequence of uniqueness of the closed orbit:

3.5. COROLLARY. *Let $P_1 \subset P_2$ be parabolic subgroups of G , $X_i = G/P_i$ the resulting complex flag manifolds, and $\pi: X_1 \rightarrow X_2$ the natural projection. Let G_0 be a real form of G , $x_i \in X_i$ such that $G_0(x_i)$ is the closed orbit on X_i . Then*

$$\pi G_0(x_1) = G_0(x_2) \quad \text{and} \quad G_0(x_1) \subset \pi^{-1}G_0(x_2).$$

Now we count dimensions, proving

3.6. THEOREM. *Let $X = G/P$ be a complex flag manifold, $x \in X$, G_0 a real form of G , and $\mathfrak{O}_x = \mathfrak{O}_\Phi$ relative to a simple root system on a τ -stable Cartan subalgebra $\mathfrak{H} \subset \mathfrak{O}_x$ of \mathfrak{g} .*

1. $\dim_{\mathbb{R}} G_0(x) \geq \dim_{\mathbb{C}} X \quad \{ = \frac{1}{2} \dim_{\mathbb{R}} X \}$.
2. *The following conditions are equivalent:*

- (2a) $\dim_{\mathbb{R}} G_0(x) = \dim_{\mathbb{C}} X$.
- (2b) $\tau\Phi^u = \Phi^u$.

(2c) $\tau\mathfrak{O}_x = \mathfrak{O}_x$, i.e. $\tau P_x = P_x$, i.e. *viewing G as a linear algebraic group defined over \mathbb{R} such that G_0 is the topological identity component of the subgroup $G_{\mathbb{R}}$ of real points, the algebraic subgroup P_x of G is def/\mathbb{R} .*

(2d) $G_0(x)$ is the closed orbit; with G def/\mathbb{R} as in (2c), some conjugate of P is def/\mathbb{R} .

(2e) X is a complex projective variety def/\mathbb{R} in such a manner that $G_0(x)$ is the set $X_{\mathbb{R}}$ of real points.

PROOF. $\dim_{\mathbb{R}} G_0(x) = 2(r - r_{\Phi}) - |\Phi^u \cap \tau\Phi^u| \geq 2(r - r_{\Phi}) - |\Phi^u| = r - r_{\Phi} = \dim_{\mathbb{C}} X$, with equality if and only if $\Phi^u = \tau\Phi^u$. This proves (1) and gives equivalence of (2a) and (2b). $\tau\Phi^u = \Phi^u$ if and only if $\tau\mathfrak{O}_{\Phi}^u = \mathfrak{O}_{\Phi}^u$ because $\tau\mathfrak{H} = \mathfrak{H}$, $\tau\mathfrak{O}_{\Phi}^u = \mathfrak{O}_{\Phi}^u$ if and only if $\tau\mathfrak{O}_{\Phi} = \mathfrak{O}_{\Phi}$ because \mathfrak{O}_{Φ} (resp. $\tau\mathfrak{O}_{\Phi}$) is the normalizer of \mathfrak{O}_{Φ}^u (resp. $\tau\mathfrak{O}_{\Phi}^u$) in \mathfrak{g} ; this gives equivalence of (2b) and (2c).

(1), the characterization of the closed orbit as the orbit of lowest dimension, and equivalence of (2a) and (2c), show that (2c) implies (2d). Conversely let $G_0(x)$ be closed and some $P_x \text{ def}/\mathbf{R}$; then $\dim_{\mathbf{R}} G_0(x) = \dim_{\mathbf{e}} X$, so $G_0(x)$ is a lowest dimensional orbit, so $G_0(x) = G_0(z)$ and we have (2a). Thus (2c) and (2d) are equivalent.

(2e) implies (2a), thus (2c). Conversely assume (2c). Then the projective variety X is def/\mathbf{R} and $G_0(x)$ is a topological component of $X_{\mathbf{R}}$. But $X_{\mathbf{R}}$ is a finite union of topological components $X_{\mathbf{R}}^{\mathcal{Q}}$, G_0 is transitive on each $X_{\mathbf{R}}^{\mathcal{Q}}$, each $X_{\mathbf{R}}^{\mathcal{Q}}$ is closed in X , and there is just one closed G_0 -orbit. Now there is just one $X_{\mathbf{R}}^{\mathcal{Q}}$, so $G_0(x) = X_{\mathbf{R}}$ and we have (2e). Thus (2c) and (2e) are equivalent. Q.E.D.

We reformulate condition (2c) of Theorem 3.6 in terms of the action of the Galois group of \mathbf{C} over \mathbf{R} on a simple root system.

3.7. COROLLARY. *Let $X = G/P$ be a complex flag manifold, $x \in X$, G_0 a real form of G , $\mathfrak{K} \subset \mathfrak{O}_x$ a τ -stable Cartan subalgebra of \mathfrak{G} , and Π a simple root system such that $\mathfrak{O}_x = \mathfrak{O}_{\Phi}$ for some $\Phi \subset \Pi$. Define $\nu = w \cdot \tau$ where w is an element of the Weyl group which sends $\tau(\Pi)$ back to Π . Then $\dim_{\mathbf{R}} G_0(x) = \dim_{\mathbf{e}}(X)$ if and only if*

- (i) $\nu\Phi = \Phi$ and
- (ii) *there is a minimal parabolic subgroup $MAN \subset G_0$ such that $\mathfrak{M}^{\mathbf{e}} + \mathfrak{A}^{\mathbf{e}} + \mathfrak{N}^{\mathbf{e}} = \mathfrak{O}_{\Psi}$ with $\Psi \subset \Phi$.*

PROOF. $\{1, \nu\}$ is the action of the Galois group of \mathbf{C} over \mathbf{R} on the simple root system Π , and (ii) is the statement that $\mathfrak{O}_x \cap \mathfrak{G}_0$ contains a minimal parabolic subalgebra of \mathfrak{G}_0 . Now (i) and (ii) just say that $\tau\mathfrak{O}_x = \mathfrak{O}_x$. Q.E.D.

We reformulate condition (2e) of Theorem 3.6 in terms of linear representations. To do that we order the weights by Π and say that a positive weight λ belongs to the subset $\Phi \subset \Pi$ if $\Phi = \{\varphi \in \Pi: \langle \lambda, \varphi \rangle = 0\}$. If λ is a positive weight then π_{λ} is the corresponding irreducible \mathbf{C} -linear representation of \mathfrak{G} , V^{λ} is the representation space, and v_{λ} denotes an arbitrary nonzero highest weight vector. We have the usual map $p: V^{\lambda} \rightarrow \text{proj}(V^{\lambda})$ onto the complex projective space determined by V^{λ} , and $\tilde{\pi}_{\lambda}$ denotes the projective representation of G on $\text{proj}(V^{\lambda})$ determined by these data. If λ belongs to Φ we note that $\mathfrak{O}_{\Phi} = \{h \in \mathfrak{G}: \pi_{\lambda}(h)(v_{\lambda}) \text{ is a multiple of } v_{\lambda}\}$, so $X = G/P_{\Phi}$ is realized as a subvariety of $\text{proj}(V^{\lambda})$ which is an orbit, $X = \tilde{\pi}_{\lambda}(G)(pv_{\lambda})$.

3.8. COROLLARY. *Let $X = G/P$ be a complex flag manifold, $x \in X$, G_0 a real form of G , $\mathfrak{K} \subset \mathfrak{O}_x$ a τ -stable Cartan subalgebra of \mathfrak{G} , and Π a simple root system such that $\mathfrak{O}_x = \mathfrak{O}_{\Phi}$ for some $\Phi \subset \Pi$. Then $\dim_{\mathbf{R}} G_0(x) = \dim_{\mathbf{C}} X$ if and only if there is a positive weight λ which belongs to Φ , such that*

- (i) V^λ has a $\pi_\lambda(\mathfrak{G}_0)$ -stable real form $V_{\mathbf{R}}^\lambda$ and
- (ii) $V_{\mathbf{R}}^\lambda$ contains a nonzero highest weight vector v_λ .

In that case, identifying $X = \pi_\lambda(G)(pv_\lambda)$, the set of real points of X is given by

$$X_{\mathbf{R}} = X \cap p(V_{\mathbf{R}}^\lambda) = \pi_\lambda(G_0)(pv_\lambda).$$

We specialize Theorem 3.6 to hermitian symmetric spaces.

3.9. COROLLARY. Let $X = G/P = G_u/K$ be a hermitian symmetric space of compact type, let $X_0 = G_0/K$ be the dual bounded symmetric domain, take the Borel embedding $X_0 \subset X$ given by $X_0 = G_0(x_0)$ where $P = P_{x_0}$ and $K = G_0 \cap P = G_u \cap P$. Let \check{S} be the Bergman-Šilov boundary of X_0 in X . Then \check{S} is the unique closed G_0 -orbit on X and is contained in the closure of every G_0 -orbit. Let $x \in \check{S}$ and let $c \in G_u$ be the Cayley transform (Korányi-Wolf [2]) carrying x_0 to x . Then the following conditions are equivalent.

- (i) $\dim_{\mathbf{R}} \check{S} = \dim_{\mathbf{e}} X$.
- (ii) X_0 is a tube domain.
- (iii) Viewing G def/ \mathbf{R} with G_0 as the identity component of $G_{\mathbf{R}}$ we have $\text{ad}(c)P$ defined over \mathbf{R} .
- (iv) X is a complex projective variety defined over \mathbf{R} in such a manner that $\check{S} = X_{\mathbf{R}}$.

PROOF. It is known [2] that \check{S} is the unique closed G_0 -orbit on the topological boundary of X_0 in X . Thus \check{S} is the unique closed G_0 -orbit on X .

Equivalence of (i), (iii) and (iv) is given by Theorem 3.6; equivalence of (i) and (ii) is given in [2]. Q.E.D.

In view of Corollary 3.9 one might hope to prove that the closed orbit is always the Bergman-Šilov boundary of an open orbit. But this is far from the case; Theorem 5.7 will show us that most open orbits do not have any nonconstant holomorphic functions.

In order to specialize Theorem 3.6 to the case where P is a Borel subgroup of G we have to be able to recognize the cases in which \mathfrak{G} has a Borel subalgebra \mathfrak{B} stable under τ . That is the case in which \mathfrak{G}_0 (resp. G_0) is a ‘‘Steinberg normal real form’’ of \mathfrak{G} (resp. of G).

Recall that \mathfrak{G}_0 (resp. G_0) is called the **Cartan normal real form** or **normal real form** of \mathfrak{G} (resp. G) when the following equivalent conditions hold.

(3.10a) G_0 has a Cartan subgroup H_0 that is split/ \mathbf{R} , i.e. that is a real vector group, i.e. such that the $\mathfrak{H}_0^{\mathbf{C}}$ -roots of \mathfrak{G} take real values on \mathfrak{H}_0 .

(3.10b) If K is a maximal compact subgroup of G_0 then the symmetric space rank of G_0/K is equal to the Lie group rank of G_0 .

(3.10c) \mathfrak{g}_0 is a direct sum of simple algebras from the list:

- type A_n : Lie algebra of $\mathbf{SL}(n+1, \mathbf{R})$
- type B_n : Lie algebra of $\mathbf{SO}^n(2n+1) = \mathbf{SO}(n, n+1)$
- type C_n : Lie algebra of $\mathbf{Sp}(n, \mathbf{R})$
- type D_n : Lie algebra of $\mathbf{SO}^n(2n) = \mathbf{SO}(n, n)$
- type G_2 : maximal compactly embedded subalgebra $A_1 \oplus A_1$
- type F_4 : maximal compactly embedded subalgebra $A_1 \oplus C_3$
- type E_6 : maximal compactly embedded subalgebra C_4
- type E_7 : maximal compactly embedded subalgebra A_7
- type E_8 : maximal compactly embedded subalgebra D_8

The Cartan normal real form of \mathfrak{g} is a Steinberg normal real form. For if H_0^c is an \mathbf{R} -split Cartan subgroup of G_0 and Δ^+ is a positive \mathfrak{K}_0^c -root system of \mathfrak{g} , then $\tau\varphi = \varphi$ for every root, so $\mathfrak{B} = \mathfrak{K}_0^c + \sum_{\Delta^+} \mathfrak{G}_\varphi$ is a τ -stable Borel subalgebra of \mathfrak{g} .

Let σ be a Cartan involution of \mathfrak{g}_0 and $\mathfrak{g}_0 = \mathfrak{K} + \mathfrak{M}$ the Cartan decomposition. Let $\mathfrak{K}_0 = \mathfrak{K}_T + \mathfrak{K}_V$ be a σ -stable Cartan subalgebra of \mathfrak{g}_0 , let $\mathfrak{K} = \mathfrak{K}_0^c$, choose a system Δ^+ of positive \mathfrak{K} -roots of \mathfrak{g} , and let $\mathfrak{B} = \mathfrak{K} + \sum_{\Delta^+} \mathfrak{G}_\varphi$. If $\tau\mathfrak{B} = \mathfrak{B}$ then we have $h \in i\mathfrak{K}_T + \mathfrak{K}_V$ such that

$$\{\varphi \in \Delta: \varphi(h) > 0\} = \Delta^+ = \{\varphi \in \Delta: \varphi(\tau h) > 0\}.$$

Then $h = h_T + h_V$, $\tau(h) = -h_T + h_V$, such that

$$\Delta^+ = \{\varphi \in \Delta^+: \varphi(h_V) > 0\}.$$

In other words \mathfrak{K}_V contains a regular element of \mathfrak{g} , and thus \mathfrak{K}_0 is a maximally split Cartan subalgebra of \mathfrak{g}_0 . Conversely, if \mathfrak{K}_V contains a regular element of \mathfrak{g} , that element defines a τ -stable positive root system Δ^+ for which $\tau\mathfrak{B} = \mathfrak{B}$.

Now we can say that \mathfrak{g}_0 (resp. G_0) is a **Steinberg normal real form** of \mathfrak{g} (resp. of G) if and only if the following equivalent conditions hold. There $\mathfrak{K}_0 = \mathfrak{J} + \mathfrak{A}$ is a maximally split Cartan subalgebra of \mathfrak{g}_0 .

(3.11a) \mathfrak{g} has a Borel subalgebra \mathfrak{B} with $\tau\mathfrak{B} = \mathfrak{B}$.

(3.11b) \mathfrak{A} contains a regular element of \mathfrak{g} .

(3.11c) Every \mathfrak{K}_0^c -root has nonzero restriction to \mathfrak{A} .

(3.11d) Let K be a maximal compact subgroup of G_0 and let \mathfrak{s} be the Satake diagram of the symmetric space G_0/K . Then \mathfrak{s} has no black vertices and \mathfrak{K} contains no simple ideal of \mathfrak{g}_0 .

(3.11e) \mathfrak{g}_0 is a direct sum of simple algebras from the list:

any type: complex simple Lie algebra

any type: Cartan normal real form

type A_{2n-1} : Lie algebra of $SU^n(2n) = SU(n, n)$

type D_n : Lie algebra of $SO^{n-1}(2n) = SO(n-1, n+1)$

type E_6 : maximal compactly embedded subalgebra $A_1 \oplus A_5$

Summarizing, we have the following special case of Theorem 3.6.

3.12. COROLLARY. Let $X = G/B$, complex flag manifold where B is a Borel subgroup of G . Let $x \in X$ and G_0 a real form of G , such that $G_0(x)$ is the closed orbit on X . Then the following conditions are equivalent:

- (i) $\dim_{\mathbb{R}} G_0(x) = \dim_{\mathbb{C}} X$.
- (ii) G is a Steinberg normal real form of G .
- (iii) \mathfrak{G}_0 satisfies (3.11e) above.

4. Open orbits: construction, covering and counting. We work with a complex flag manifold $X = G/P$, a point $x \in X$, a real form G_0 of G , conjugation τ of G over G_0 , a τ -stable Cartan subalgebra $\mathfrak{K} \subset \mathfrak{P}_x$ of \mathfrak{g} , and a system Π of simple roots such that $\mathfrak{P}_x = \mathfrak{P}_{\Phi}$ for some $\Phi \in \Pi$. Theorem 2.12 says that $G_0(x)$ is an open G_0 -orbit on X if and only if $\Phi^u \cap \tau\Phi^u$ is empty.

4.1. LEMMA. Let \mathfrak{K}_0 be a Cartan subalgebra of \mathfrak{G}_0 . Choose a maximal compact subgroup K of G_0 , and take the Cartan decomposition $\mathfrak{G}_0 = \mathfrak{K} + \mathfrak{M}$, such that

$$(4.2) \quad \mathfrak{K}_0 = \mathfrak{K}_T + \mathfrak{K}_V, \quad \mathfrak{K}_T = \mathfrak{K}_0 \cap \mathfrak{K}, \quad \mathfrak{K}_V = \mathfrak{K}_0 \cap \mathfrak{M},$$

decomposition of \mathfrak{K}_0 into toral and vector parts. Then the following conditions are equivalent:

- (i) \mathfrak{K}_T is a Cartan subalgebra of \mathfrak{K} .
- (ii) \mathfrak{K}_T contains a regular element of \mathfrak{G} .
- (iii) τ sends some Weyl chamber of \mathfrak{K} to its negative.
- (iv) There is a system Δ^+ of positive \mathfrak{K}_0^c -roots, $\Delta^- = -\Delta^+$, such that $\tau\Delta^+ = \Delta^-$.

PROOF. (i) implies (ii) because \mathfrak{K} contains a regular element of \mathfrak{G} . Let $h_0 \in \mathfrak{K}_T$ be a regular element of \mathfrak{g} . Then \mathfrak{K}_0 is the centralizer of h_0 in \mathfrak{G}_0 , so \mathfrak{K}_T is its own centralizer in \mathfrak{K} ; this shows that (ii) implies (i).

(iii) and (iv) are equivalent; for if $\mathfrak{D} \subset i\mathfrak{K}_T + \mathfrak{K}_V$ is a Weyl chamber and $\Delta^+ = \{\varphi \in \Delta: \varphi > 0 \text{ on } \mathfrak{D}\}$ is the corresponding positive root system, then $\tau(\mathfrak{D}) = -\mathfrak{D}$ if and only if $\tau(\Delta^+) = \Delta^-$.

(i) implies (iv); for if $h_0 \in \mathfrak{K}_T$ is a regular element we have $\tau(ih_0) = -ih_0$ and $\{\varphi \in \Delta: \varphi(ih_0) > 0\}$ is a positive root system. Conversely

if \mathfrak{K}_T contains no regular element we have a root φ such that $\varphi(\mathfrak{K}_T) = 0$. Then $\varphi \in \mathfrak{K}_V$, so $\tau(\varphi) = \varphi$, and $\tau\Delta^+ = \Delta^-$ is impossible for every positive root system. Thus (iv) implies (i). Q.E.D.

4.3. DEFINITION. A Cartan subalgebra $\mathfrak{K}_0 \subset \mathfrak{G}_0$ is **maximally compact** if it satisfies the conditions of Lemma 4.1, is **compact** if moreover $\mathfrak{K}_0 = \mathfrak{K}_T$ in Lemma 4.1, i.e. if $\text{rank } K = \text{rank } G_0$.

4.4. LEMMA. Any two maximally compact Cartan subalgebras of \mathfrak{G}_0 are conjugate.

PROOF. Let \mathfrak{K}_0 and \mathfrak{K}'_0 be maximally compact Cartan subalgebras of \mathfrak{G}_0 . We have maximal compact subgroups $K, K' \subset G_0$, Cartan decompositions $\mathfrak{K} + \mathfrak{M} = \mathfrak{G}_0 = \mathfrak{K}' + \mathfrak{M}'$, and expressions $\mathfrak{K}_0 = \mathfrak{K}_T + \mathfrak{K}_V$ and $\mathfrak{K}'_0 = \mathfrak{K}'_T + \mathfrak{K}'_V$ as in (4.2). By hypothesis \mathfrak{K}_T is a Cartan subalgebra of \mathfrak{K} and \mathfrak{K}'_T is a Cartan subalgebra of \mathfrak{K}' .

G_0 has an element g such that $\text{ad}(g)\mathfrak{K}' = \mathfrak{K}$, so we may replace \mathfrak{K}'_0 by its conjugate $\text{ad}(g)\mathfrak{K}'_0$ and assume $\mathfrak{K} = \mathfrak{K}'$. K has an element k such that $\text{ad}(k)\mathfrak{K}'_T = \mathfrak{K}_T$, so we may further replace \mathfrak{K}'_0 by its conjugate $\text{ad}(k)\mathfrak{K}'_0$ and assume $\mathfrak{K}'_T = \mathfrak{K}_T$. Now $\mathfrak{K}_0 = \mathfrak{K}_T + \mathfrak{K}_V$ and $\mathfrak{K}'_0 = \mathfrak{K}_T + \mathfrak{K}'_V$. Let $h_0 \in \mathfrak{K}_T$ be a regular element of \mathfrak{g} . Then $\mathfrak{K}_0 = \mathfrak{L} = \mathfrak{K}'_0$ where \mathfrak{L} is the centralizer of h_0 in \mathfrak{G}_0 . Q.E.D.

4.5. THEOREM. Let $X = G/P$ be a complex flag manifold, $x \in X$ and G_0 a real form of G . Then the orbit $G_0(x)$ is open in X if and only if there exist

- (i) a maximally compact Cartan subalgebra $\mathfrak{K}_0 \subset \mathfrak{G}_0 \cap \mathfrak{P}_x$ of \mathfrak{G}_0 , and
- (ii) a system Π of simple $\mathfrak{K}_0^\mathbb{C}$ -roots such that $\tau\Delta^+ = \Delta^-$ and $\mathfrak{P}_x = \mathfrak{P}_\Phi$ for some $\Phi \subset \Pi$.

If $P_1 \subset P_2$ are parabolic subgroups of G , $X_i = G/P_i$ the resulting complex flag manifolds, $x_i \in X_i$ and $\pi: X_1 \rightarrow X_2$ the natural projection, then

- (1) if $G_0(x_1)$ is an open orbit on X_1 then $\pi(G_0(x_1)) = G_0(\pi x_1)$ is an open orbit on X_2 , and
- (2) if $G_0(x_2)$ is an open orbit on X_2 then $\pi^{-1}(G_0(x_2))$ contains an open G_0 -orbit on X_1 .

PROOF. We first prove the assertions on $\pi: X_1 \rightarrow X_2$. If $G_0(x_2)$ is open in X_2 then $\pi^{-1}(G_0(x_2))$ is open in X_1 because π is continuous. As G_0 commutes with π , $\pi^{-1}(G_0(x_2))$ is a (necessarily finite) union of G_0 -orbits, so it contains an orbit of full dimension in X_1 ; that orbit is necessarily open. This proves assertion (2). If $G_0(x_1)$ is open in X_1 then $\pi(G_0(x_1)) = G_0(\pi x_1)$ is open in X_2 because π is an open map. This proves assertion (1).

Suppose for the moment that P is a Borel subgroup of G and let $\mathfrak{K} \subset \mathfrak{P}_x$ be a τ -stable Cartan subalgebra of \mathfrak{g} . There is just one simple

root system Π such that $\mathcal{O}_x = \mathcal{O}_\Phi$ with $\Phi \subset \Pi$; Φ is empty and Π is characterized by $\Phi^u = \Delta^+$. If $G_0(x)$ is open, so $\Phi^u \cap \tau\Phi^u$ is empty by Theorem 2.12, then we have $\tau\Delta^+ = \Delta^-$, and $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{G}_0$ is a maximally compact Cartan subalgebra of \mathcal{G}_0 by Lemma 4.1. Conversely, if $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{G}_0$ is maximally compact and $\tau\Delta^+ = \Delta^-$, then $\Phi^u \cap \tau\Phi^u$ is empty, so $G_0(x)$ is open in X by Theorem 2.12. The first statement is proved for the case where P is a Borel subgroup of G .

Now in general choose a Borel subgroup $B \subset P$ of G , let $Y = G/B$ be the complex flag manifold, and let $\pi: Y \rightarrow X$ be the natural projection. First suppose $G_0(x)$ open in X . Then assertion (2), which was already proved, provides an element $y \in \pi^{-1}(x)$ such that $G_0(y)$ is open in Y . Note $\mathcal{B}_y \subset \mathcal{O}_x$. Choose a τ -stable Cartan subalgebra $\mathcal{K} \subset \mathcal{B}_y$ of \mathcal{G} and the simple root system Π such that $\mathcal{B}_y = \mathcal{O}_\Phi$, Φ empty. As was seen, $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{G}_0$ is a maximally compact Cartan subalgebra of \mathcal{G}_0 , and $\tau\Delta^+ = \Delta^-$, because $G_0(y)$ is open in Y . Now $\mathcal{O}_x = \mathcal{O}_\Psi$ for some $\Psi \subset \Pi$ because $\mathcal{B}_y \subset \mathcal{O}_x$. Thus we have (i) and (ii). Conversely suppose that there is a maximally compact Cartan subalgebra $\mathcal{K}_0 \subset \mathcal{O}_x$ of \mathcal{G}_0 , a system Π of simple \mathcal{K}_0^e -roots such that $\tau\Delta^+ = \Delta^-$, and a subset $\Psi \subset \Pi$ such that $\mathcal{O}_x = \mathcal{O}_\Psi$. $\tau\Delta^+ = \Delta^-$ says that $\Delta^+ \cap \tau\Delta^+$ is empty. $\Psi \subset \Pi$ says that $\Psi^u \subset \Delta^+$, so now $\Psi^u \cap \tau\Psi^u$ is empty. Theorem 2.12 now implies that $G_0(x)$ is open in X . Q.E.D.

Now we set out to construct and count the open G_0 -orbits on $X = G/P$. To do this we choose a Borel subgroup $B \subset P$ and let $\pi: Y \rightarrow X$ be the projection, $Y = G/B$. First we construct and count the open G_0 -orbits on Y , and then we carry the result down to X .

4.6. THEOREM. *Let $Y = G/B$, complex flag manifold where B is a Borel subgroup of G . Let G_0 be a real form of G and choose a maximally compact Cartan subalgebra $\mathcal{K}_0 = \mathcal{K}_T + \mathcal{K}_V$ of \mathcal{G}_0 . Let $\{\mathcal{D}_1, \dots, \mathcal{D}_m\}$ be the chambers in $i\mathcal{K}_T$ cut out⁴ by the \mathcal{K}_0^e -roots of \mathcal{G} , and let W_K denote the Weyl group of K for the Cartan subalgebra \mathcal{K}_T of \mathcal{K} .*

1. *Let $1 \leq \alpha \leq m$, choose $h_\alpha \in \mathcal{D}_\alpha$, let Π_α be the system of simple \mathcal{K}_0^e -roots of \mathcal{G} for the positive root system $\Delta_\alpha^+ = \{\varphi \in \Delta: \varphi(h_\alpha) > 0\}$, let $\mathcal{B}_\alpha = \mathcal{K}_0^e + \sum_{\Delta_\alpha^+} \mathcal{G}_\varphi$ be the corresponding Borel subalgebra of \mathcal{G} and let $y_\alpha \in Y$ be the element such that $\mathcal{B}_{y_\alpha} = \mathcal{B}_\alpha$. Then the open G_0 -orbits on Y are just the $G_0(y_\alpha)$, and $G_0(y_\alpha) = G_0(y_\beta)$ if and only if some element of W_K sends \mathcal{D}_α to \mathcal{D}_β .*

2. *W_K acts simply on $\{\mathcal{D}_1, \dots, \mathcal{D}_m\}$.*

⁴ Given a root φ , $\{h \in i\mathcal{K}_T: \varphi(h) = 0\}$ is a subspace of codimension ≥ 1 in $i\mathcal{K}_T$. The union of these subspaces, as φ ranges over the root system Δ , is the intersection of $i\mathcal{K}_T$ with the "diagram" in $i\mathcal{K}_T + \mathcal{K}_V$. The \mathcal{D}_α are the components of the complement of this intersection in $i\mathcal{K}_T$, i.e. of the set of regular elements of \mathcal{G} contained in $i\mathcal{K}_T$.

3. *There are precisely $m/|W_K|$ distinct open G_0 -orbits on Y .*

PROOF. Let $G_0(y)$ be an open orbit on Y . Theorem 4.5(i) gives us a maximally compact Cartan subalgebra $\mathfrak{K}'_0 \subset \mathfrak{B}_y$ of \mathfrak{G}_0 and Lemma 4.4 gives us an element $g \in G_0$ such that $\text{ad}(g)\mathfrak{K}'_0 = \mathfrak{K}_0$. Replacing y by $g(y)$, $G_0(y)$ is not changed but we may assume $\mathfrak{K}_0 \subset \mathfrak{B}_y$. Theorem 4.5(ii) provides a simple root system Π , and the corresponding system Δ^+ of positive roots, such that $\tau\Delta^+ = \Delta^-$ and $\mathfrak{B}_y = \mathfrak{K}_0^e + \sum_{\Delta^+} \mathfrak{G}_\varphi$. Let $h \in i\mathfrak{K}_T + \mathfrak{K}_V$ be a regular element such that $\Delta^+ = \{\varphi \in \Delta: \varphi(h) > 0\}$. Decompose $h = h_T + h_V$, $h_T \in i\mathfrak{K}_T$ and $h_V \in \mathfrak{K}_V$, so $\varphi(h) = \varphi(h_T) + \varphi(h_V)$ and $(\tau\varphi)(h) = \varphi(\tau h) = -\varphi(h_T) + \varphi(h_V)$ for $\varphi \in \Delta$. Now let $\varphi \in \Delta^+$ and use $\tau\Delta^+ = \Delta^-$; so $\varphi(h_T) + \varphi(h_V) > 0 > -\varphi(h_T) + \varphi(h_V)$, which implies $\varphi(h_T) > 0$. Thus $\Delta^+ = \Delta_\alpha^+$, so $\mathfrak{B}_y = \mathfrak{B}_\alpha$, so $y = y_\alpha$, so $G_0(y) = G_0(y_\alpha)$, for the index α with $h_T \in \mathfrak{D}_\alpha$. This proves that every open G_0 -orbit on Y is one of the $G_0(y_\alpha)$. Conversely, $\tau(h_\alpha) = -h_\alpha$, which implies $\tau\Delta_\alpha^+ = \Delta_\alpha^-$, so $G_0(y_\alpha)$ is an open orbit.

Let $w \in W_K$ send \mathfrak{D}_α to \mathfrak{D}_β . Represent $w = \text{ad}(k)|\mathfrak{K}_0^e$, $k \in K$. Now $\text{ad}(k)\mathfrak{B}_\alpha = \mathfrak{B}_\beta$, i.e. $k(y_\alpha) = y_\beta$, so $G_0(y_\beta) = G_0(ky_\alpha) = (G_0k)(y_\alpha) = G_0(y_\alpha)$. Conversely suppose $G_0(y_\alpha) = G_0(y_\beta)$, so $g(y_\alpha) = y_\beta$ for some $g \in G_0$. \mathfrak{K}_0 is both the centralizer of y_α in \mathfrak{G}_0 and the centralizer of y_β in \mathfrak{G}_0 ; now $\text{ad}(g)\mathfrak{K}_0 = \mathfrak{K}_0$ and $\text{ad}(g)\mathfrak{B}_\alpha = \mathfrak{B}_\beta$. Thus g represents an element w of the Weyl group W_{G_0} of G_0 relative to \mathfrak{K}_0 , such that $w(\mathfrak{D}_\alpha) = \mathfrak{D}_\beta$. The analytic subgroup for \mathfrak{K}_0 in G_0 has finite index in its normalizer, for the latter is algebraic and thus has only finitely many components. It follows that $g = kv$ where $k \in K$ represents w and $v \in \exp(\mathfrak{K}_V)$. In other words $w \in W_K$. We have proved that $G_0(y_\alpha) = G_0(y_\beta)$ if and only if an element of W_K sends \mathfrak{D}_α to \mathfrak{D}_β .

Let $w \in W_K$ such that $w(\mathfrak{D}_\alpha) = \mathfrak{D}_\alpha$. As $W_K \subset W_G$, we have $w \in W_G$. $\mathfrak{D}_\alpha = \mathfrak{D} \cap i\mathfrak{K}_T$ for some Weyl chamber $\mathfrak{D} \subset i\mathfrak{K}_T + \mathfrak{K}_V$; as \mathfrak{D}_α is non-empty now $w(\mathfrak{D}) = \mathfrak{D}$. That says $w = 1$. Now W_K acts simply on $\{\mathfrak{D}_1, \dots, \mathfrak{D}_m\}$.

Now we can count the open G_0 -orbits. They are the $G_0(y_\alpha)$, $1 \leq \alpha \leq m$, and the action of W_K partitions them into sets of $|W_K|$ elements under the relation of W_K -equivalence. Thus there are $m/|W_K|$ distinct open orbits. Q.E.D.

If $\text{rank } K = \text{rank } G_0$, i.e. if \mathfrak{K}_0 is a compact Cartan subalgebra of \mathfrak{G}_0 in the notation of Theorem 4.6, then $\mathfrak{K}_0 = \mathfrak{K}_T$ and the \mathfrak{D}_α are the Weyl chambers. Then the number m of \mathfrak{D}_α is the order $|W_G|$ of the Weyl group of G , so part 3 of Theorem 4.6 reduces to

4.7. COROLLARY. *Let $Y = G/B$, complex flag manifold where B is a Borel subgroup of G . Let G_0 be a real form of G whose maximal compact subgroup K satisfies $\text{rank } K = \text{rank } G_0$. Then there are precisely $|W_G|/|W_K|$ distinct open G_0 -orbits on Y .*

On the other hand, if $\text{rank } K < \text{rank } G_0$, there may be Weyl chambers in $i\mathfrak{C}_T + \mathfrak{C}_V$ which do not meet $i\mathfrak{C}_T$, so the number of \mathfrak{D}_α may be less than $|W_G|$. For example, if $G = \text{SL}(3, \mathbf{C})$ and $G_0 = \text{SL}(3, \mathbf{R})$, then $|W_G| = 6$, so there are 6 Weyl chambers, but 4 of them are not sent to their negatives by τ ; thus $m = 2$, $|W_K| = |W_{\text{so}(\mathfrak{g})}| = 2$, and so there is just one open G_0 -orbit on G/B . This trick for finding m works in general:

4.8. COROLLARY. *Let $Y = G/B$, complex flag manifold where B is a Borel subgroup of G . Let G_0 be a real form of G , let K be a maximal compact subgroup of G_0 , let σ denote the involutive automorphism of \mathfrak{g}_0 with fixed point set \mathfrak{K} , and let $\mathfrak{C}_0 = \mathfrak{C}_T + \mathfrak{C}_V$ be a maximally compact Cartan subalgebra of \mathfrak{g}_0 such that $\mathfrak{C}_T \subset \mathfrak{K}$. Define*

- W_G : Weyl group of G for the Cartan subalgebra \mathfrak{C}_0 ,
- W_K : Weyl group of K for the Cartan subalgebra \mathfrak{C}_T ,
- $W_G^{\mathfrak{C}_0}$: $\{w \in W_G : w(\mathfrak{C}_0) = \mathfrak{C}_0\}$, subgroup of W_G .

Then $W_G^{\mathfrak{C}_0}$ is simply transitive on the set of Weyl chambers which meet $i\mathfrak{C}_T$ (i.e. which are sent to their negatives by τ), $W_G^{\mathfrak{C}_0}$ is the centralizer of $\sigma|_{\mathfrak{C}_T + \mathfrak{C}_V}$ in W_G , and G_0 has precisely $|W_G^{\mathfrak{C}_0}| / |W_K|$ distinct open orbits on Y .

PROOF. Let \mathfrak{D} be a Weyl chamber which meets $i\mathfrak{C}_T$, i.e. such that $\tau\mathfrak{D} = -\mathfrak{D}$. Note that $\sigma = -\tau$ on the ambient space $i\mathfrak{C}_T + \mathfrak{C}_V$ of the Weyl chambers. Let $w \in W_G$ and notice $-w(\mathfrak{D}) = w(\tau\mathfrak{D})$, so $\tau(w\mathfrak{D}) = -w\mathfrak{D}$ if and only if $w\mathfrak{D} = \tau^{-1}w\tau\mathfrak{D}$, i.e. if and only if $w\tau = \tau w$, i.e. if and only if $w\sigma = \sigma w$, i.e. if and only if w preserves $i\mathfrak{C}_T$ and \mathfrak{C}_V , i.e. if and only if $w \in W_G^{\mathfrak{C}_0}$. Now our assertions follow from part 3 of Theorem 4.6. Q.E.D.

Now we push Theorem 4.6 down to a general complex flag manifold.

4.9. THEOREM. *Let $X = G/P$ be a complex flag manifold, let $B \subset P$ be a Borel subgroup of G , let $Y = G/B$, and let $\pi : Y \rightarrow X$ be the natural projection. Let G_0 be a real form of G , let $x \in X$ such that $G_0(x)$ is open in X , and let $\mathfrak{C}_0 \subset \mathfrak{g}_0 \cap \mathfrak{P}_x$ be a maximally compact Cartan subalgebra of \mathfrak{g}_0 . Decompose $\mathfrak{P}_x = \mathfrak{P}_x^r + \mathfrak{P}_x^u$ relative to $\mathfrak{C} = \mathfrak{C}_0$ and define $\mathfrak{P}_x^{-u} = \sum_{-\mathfrak{P}_x^u} \mathfrak{G}_\mathfrak{P}$.*

1. *Let $y \in \pi^{-1}(x)$. Then $G_0(y)$ is open in Y if and only if $\mathfrak{B}_y = \mathfrak{B}' + (\mathfrak{P}_x^r \cap \tau\mathfrak{P}_x^{-u}) + \mathfrak{P}_x^u$ where \mathfrak{B}' is a Borel subalgebra of $\mathfrak{P}_x^r \cap \tau\mathfrak{P}_x^r = (\mathfrak{P}_x \cap \tau\mathfrak{P}_x)^r$ such that $\mathfrak{B}' \cap \tau\mathfrak{B}' = \mathfrak{C}$.*

2. *The open G_0 -orbits in Y which lie over $G_0(x)$ are in one to one correspondence with the open $(P_x^r \cap \tau P_x^r)$ -orbits on $(P_x^r \cap \tau P_x^r) / (\text{Borel})$.*

3. *The open G_0 -orbits on X are enumerated by the elements of the double coset space $W_K \backslash W_G^{\mathfrak{C}_0} / W_{P_x^r \cap \tau P_x^r}^{\mathfrak{C}_0}$.*

PROOF. Let $y \in \pi^{-1}(x)$ with $G_0(y)$ open in Y . We may assume $\mathcal{H} \subset \mathcal{B}_y$. Then $\mathcal{B}_y = \mathcal{B}'' + \mathcal{O}_x^u = \mathcal{H} + \sum_{\Delta^+} \mathcal{G}_\varphi$ where Δ^+ is a positive root system such that $\Phi^u \subset \Delta^+$ and $\tau\Delta^+ = \Delta^-$, and where \mathcal{B}'' is a Borel subalgebra of \mathcal{O}_x^r . Decompose \mathcal{O}_x^r as the sum of $(\mathcal{O}_x^r \cap \tau\mathcal{O}_x^r)$, $(\mathcal{O}_x^r \cap \tau\mathcal{O}_x^u)$ and $(\mathcal{O}_x^r \cap \tau\mathcal{O}_x^{-u})$. $\Phi^u \subset \Delta^+$ and $\tau\Delta^+ = \Delta^-$ imply $\tau\Phi^u \subset \Delta^-$, so $\mathcal{B}'' = \mathcal{B}' + (\mathcal{O}_x^r \cap \tau\mathcal{O}_x^{-u})$ where \mathcal{B}' is a Borel subalgebra of $\mathcal{O}_x^r \cap \tau\mathcal{O}_x^r$. Now $\tau\Delta^+ = \Delta^-$ implies $\mathcal{B}' \cap \tau\mathcal{B}' = \mathcal{H}$.

Conversely let $y \in \pi^{-1}(x)$ with $\mathcal{B}_y = \mathcal{B}' + (\mathcal{O}_x^r \cap \tau\mathcal{O}_x^{-u}) + \mathcal{O}_x^u$ where \mathcal{B}' is a Borel subalgebra of $\mathcal{O}_x^r \cap \tau\mathcal{O}_x^r$ such that $\mathcal{B}' \cap \tau\mathcal{B}' = \mathcal{H}$. Then $\mathcal{B}_y \cap \tau\mathcal{B}_y = \mathcal{H}$, so $G_0(y)$ is open in Y . This completes the proof of (1).

The number of distinct Borel subalgebras $\mathcal{B}' + (\mathcal{O}_x^r \cap \tau\mathcal{O}_x^{-u}) + \mathcal{O}_x^u$, $\mathcal{B}' \subset \mathcal{O}_x^r \cap \tau\mathcal{O}_x^r$ Borel subalgebra, $\mathcal{B}' \cap \tau\mathcal{B}' = \mathcal{H}$, is the number of chambers cut out on $i\mathcal{H}_T$ by the \mathcal{H} -roots of $\mathcal{O}_x^r \cap \tau\mathcal{O}_x^r = (\mathcal{O}_x \cap \tau\mathcal{O}_x)^r$. Now, as in Theorem 4.6(3), the number of open G_0 -orbits on Y over $G_0(x)$ is the number of open $(P_x^r \cap \tau P_x^r)$ -orbits on $(P_x^r \cap \tau P_x^r)/(\text{Borel})$.

Finally we count the open G_0 -orbits on X . Corollary 4.8 identifies the coset space $W_K \backslash W_{\mathcal{G}^{\mathcal{H}_0}}$ with the open G_0 -orbits on Y as follows. Choose a Weyl chamber $\mathcal{D} \subset i\mathcal{H}_T + \mathcal{H}_V$ such that $\tau\mathcal{D} = -\mathcal{D}$. Then all other such Weyl chambers are given by the $w(\mathcal{D})$, $w \in W_{\mathcal{G}^{\mathcal{H}_0}}$; and the Borel subalgebras specified by $w_1(\mathcal{D})$ and $w_2(\mathcal{D})$, $w_i \in W_{\mathcal{G}^{\mathcal{H}_0}}$, give the same open G_0 -orbit if and only if $w_1 = w_K w_2$ for some $w_K \in W_K$. On the other hand, if \mathcal{D} specifies one of the $\mathcal{B}' + (\mathcal{O}_x^r \cap \tau\mathcal{O}_x^{-u}) + \mathcal{O}_x^u$ of (1), all other such \mathcal{D} are given by the $w(\mathcal{D})$, $w \in W_{P_x^r \cap \tau P_x^r}$; this follows from (1) and Corollary 4.8. Now (3) follows. Q.E.D.

5. Open orbits: coset space structure and holomorphic functions.

In order to study open orbits as coset spaces of the real group, and in order to study holomorphic functions on open orbits, we need a remark on compact subvarieties:

5.1. LEMMA. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , $x \in X$, such that $G_0(x)$ is open in X , and K a maximal compact subgroup of G_0 such that $\mathcal{G}_0 \cap \mathcal{O}_x$ contains a maximally compact Cartan subalgebra $\mathcal{H}_0 = \mathcal{H}_T + \mathcal{H}_V$ of \mathcal{G}_0 with $\mathcal{H}_T \subset \mathcal{H}$. Then $K(x)$ is a compact complex submanifold of $G_0(x)$.*

PROOF. Let $B \subset P$ be a Borel subgroup of G , let $Y = G/B$ and $\pi: Y \rightarrow X$ the natural projection, and choose $y \in \pi^{-1}(x)$ such that $G_0(y)$ is open in Y and $\mathcal{H}_0 = \mathcal{B}_y \cap \tau\mathcal{B}_y$. Now $\pi(K(y)) = K(x)$, and both $K(y)$ and $K(x)$ are compact real submanifolds. If $K(y)$ is a complex submanifold of $G_0(y)$, then its every tangent space $K(y)_{k(y)}$ is stable under the almost complex structure operator on Y , so $K(x)_{k(x)}$ is also stable for all $k \in K$, proving that $K(x)$ is a complex submanifold of X . Now the lemma is reduced to the case where P is a Borel subgroup of G .

Let P be a Borel subgroup. Then $\mathcal{O}_x \cap \tau\mathcal{O}_x = \mathfrak{K}_0^c$ because $G_0(x)$ is open in X . At x the $(\sqrt{-1})$ -eigenspace of the almost complex structure is the holomorphic tangent space $\sum_{\Delta^+} \mathfrak{G}_{-\varphi}$, and the $(-\sqrt{-1})$ -eigenspace is the antiholomorphic tangent space $\sum_{\Delta^+} \mathfrak{G}_{\varphi}$, where $\mathcal{O}_x = \mathfrak{K}_0^c + \sum_{\Delta^+} \mathfrak{G}_{\varphi}$. Now choose $h_0 \in i\mathfrak{K}_T^c$ such that $\Delta^+ = \{\varphi \in \Delta : \varphi(h_0) > 0\}$. Decompose $\mathfrak{K} = \mathfrak{K}_T + \mathfrak{F}$ where \mathfrak{F} is the orthocomplement under the Killing form of \mathfrak{K} . Now \mathfrak{F}^c is the sum of the \mathfrak{K}^c -root spaces on \mathfrak{K}^c , \mathfrak{F}^c gives the complexified tangent space to $K(x)$ at x , and \mathfrak{F} is the sum of its intersections with the holomorphic and antiholomorphic tangent spaces. It follows (look at h_0) that the real tangent space to $K(x)$ at x , represented by \mathfrak{F} , is stable under the almost complex structure. Thus $K(x)$ is a complex submanifold. Q.E.D.

To describe open orbits as coset spaces we need some notation. Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , $x \in X$, and $\mathfrak{K} \subset \mathcal{O}_x$ a τ -stable Cartan subalgebra of \mathfrak{G} . For a subset Φ of a simple \mathfrak{K} -root system Π we have $\mathcal{O}_x = \mathcal{O}_{\Phi}$. Write \mathcal{O}_x^r for \mathcal{O}_{Φ}^r and \mathcal{O}_x^u for \mathcal{O}_{Φ}^u . Recall, from Lemma 2.10, that $\mathcal{O}_x \cap \tau\mathcal{O}_x$ is the semidirect sum $(\mathcal{O}_x \cap \tau\mathcal{O}_x)^r + (\mathcal{O}_x \cap \tau\mathcal{O}_x)^u$ where $(\mathcal{O}_x \cap \tau\mathcal{O}_x)^r = \mathcal{O}_x^r \cap \tau\mathcal{O}_x^r$ and $(\mathcal{O}_x \cap \tau\mathcal{O}_x)^u = (\mathcal{O}_x^r \cap \tau\mathcal{O}_x^u) + (\mathcal{O}_x^u \cap \tau\mathcal{O}_x^r)$. We define real forms of $(\mathcal{O}_x \cap \tau\mathcal{O}_x)$, $(\mathcal{O}_x \cap \tau\mathcal{O}_x)^r$ and $(\mathcal{O}_x \cap \tau\mathcal{O}_x)^u$ in the obvious manner, by

$$\begin{aligned}
 (5.2a) \quad & (\mathcal{O}_x \cap \tau\mathcal{O}_x)_0 = (\mathcal{O}_x \cap \tau\mathcal{O}_x) \cap \mathfrak{G}_0, \quad \text{real form of } \mathcal{O}_x \cap \tau\mathcal{O}_x; \\
 (5.2b) \quad & (\mathcal{O}_x \cap \tau\mathcal{O}_x)_0^r = (\mathcal{O}_x^r \cap \tau\mathcal{O}_x^r) \cap \mathfrak{G}_0, \quad \text{real form of } (\mathcal{O}_x \cap \tau\mathcal{O}_x)^r; \\
 & (\mathcal{O}_x \cap \tau\mathcal{O}_x)_0^u = \{(\mathcal{O}_x^r \cap \tau\mathcal{O}_x^u) + (\mathcal{O}_x^u \cap \tau\mathcal{O}_x^r)\} \cap \mathfrak{G}_0 \\
 (5.2c) \quad & + (\mathcal{O}_x^u \cap \tau\mathcal{O}_x^u) \cap \mathfrak{G}_0, \quad \text{real form of } (\mathcal{O}_x \cap \tau\mathcal{O}_x)^u.
 \end{aligned}$$

Thus $(\mathcal{O}_x \cap \tau\mathcal{O}_x)_0 = (\mathcal{O}_x \cap \tau\mathcal{O}_x)_0^r + (\mathcal{O}_x \cap \tau\mathcal{O}_x)_0^u$ semidirect sum of reductive and nilpotent parts. Rather than proliferate notation, we denote

$$(5.3a) \quad (P_x \cap \tau P_x)_0: \text{ analytic subgroup of } G_0 \text{ for } (\mathcal{O}_x \cap \tau\mathcal{O}_x)_0,$$

$$(5.3b) \quad (P_x \cap \tau P_x)_0^r: \text{ analytic subgroup of } G_0 \text{ for } (\mathcal{O}_x \cap \tau\mathcal{O}_x)_0^r,$$

$$(5.3c) \quad (P_x \cap \tau P_x)_0^u: \text{ analytic subgroup of } G_0 \text{ for } (\mathcal{O}_x \cap \tau\mathcal{O}_x)_0^u.$$

These are all closed connected subgroups of G_0 , and $(P_x \cap \tau P_x)_0 = (P_x \cap \tau P_x)_0^r \cdot (P_x \cap \tau P_x)_0^u$ semidirect product of reductive and unipotent parts. By definition, $(P_x \cap \tau P_x)_0$ is the identity component of the isotropy subgroup $G_0 \cap P_x$ of G_0 at x .

5.4. THEOREM. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , $x \in X$, such that $G_0(x)$ is open in X . Then the open orbit $G_0(x)$ is simply connected and G_0 has connected isotropy subgroup $(P_x \cap \tau P_x)_0$ at x .*

PROOF. We work with a τ -stable Cartan subalgebra $\mathfrak{K} \subset \mathfrak{P}_x$ of \mathfrak{g} such that $\mathfrak{K}_0 = \mathfrak{K} \cap \mathfrak{G}_0$ is a maximally compact Cartan subalgebra of \mathfrak{G}_0 . Now $\mathfrak{P}_x = \mathfrak{P}_\mathfrak{K}$ as usual. Let $\mathfrak{K}_0 = \mathfrak{K}_T + \mathfrak{K}_V$, $\mathfrak{K}_T \subset \mathfrak{K}$, where K is a maximal compact subgroup of G_0 . For the moment let L denote the isotropy subgroup of G_0 at x , so $L \cap K$ is the isotropy subgroup of K . Lemma 5.1 gives K -invariant complex structure on $K/(L \cap K)$ via its realization as $K(x)$. As $\text{rank}(L \cap K) = \text{rank } K$ because $\mathfrak{K}_T \subset \mathfrak{L}$, it follows [14] that $L \cap K$ is connected and $K/(L \cap K)$ is simply connected. But $G_0(x) \cong G_0/L$ has a deformation retraction back onto $K/(L \cap K)$; thus $G_0(x)$ is simply connected and it follows that L is connected. Q.E.D.

Let $X = G/P$ be a complex flag manifold and G_0 a real form of G . Then $\mathfrak{G}_0 = \mathfrak{G}_0^{(1)} \oplus \dots \oplus \mathfrak{G}_0^{(t)}$, direct sum of simple ideals, and we have

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}^{(1)} \oplus \dots \oplus \mathfrak{g}^{(t)} \quad \text{where } \mathfrak{g}^{(i)} = \mathfrak{g}_0^{(i)e}, \\ \mathfrak{P} &= \mathfrak{P}^{(1)} \oplus \dots \oplus \mathfrak{P}^{(t)} \quad \text{where } \mathfrak{P}^{(i)} = \mathfrak{P} \cap \mathfrak{g}^{(i)}, \\ X &= X_1 \times \dots \times X_t \quad \text{where } X_i = G^{(i)}/P^{(i)}. \end{aligned}$$

Let $x \in X$, $x = (x_1, \dots, x_t)$ with $x_i \in X_i$. Then

$$G_0(x) = G_0^{(1)}(x_1) \times \dots \times G_0^{(t)}(x_t), \quad G^{(i)}(x_i) \subset X_i.$$

Of course $G_0(x)$ is open in X if and only if $G_0^{(i)}(x_i)$ is open in X_i for $1 \leq i \leq t$. Now consider the conditions

(5.5a) $(P_{x_i}^{(i)} \cap \tau P_{x_i}^{(i)})_0$ is compact; hence contained in a maximal compact subgroup $K^{(i)}$ of $G_0^{(i)}$,

(5.5b) $G_0^{(i)}/K^{(i)}$ is a hermitian symmetric coset space,

(5.5c) $G_0^{(i)}/(P_{x_i}^{(i)} \cap \tau P_{x_i}^{(i)})_0 \rightarrow G_0^{(i)}/K^{(i)}$ is holomorphic for one of the two invariant complex structures on $G_0^{(i)}/K^{(i)}$.

5.6. DEFINITION. Let $G_0(x)$ be open in X . Then the **bounded symmetric domain subordinate to the orbit** $G_0(x)$ is the coset space

$$D(G_0, x) = G_0^{(1)}/L^{(1)} \times \dots \times G_0^{(t)}/L^{(t)}$$

where

(i) if the conditions (5.5) all hold, then $L^{(i)} = K^{(i)}$ and $G_0^{(i)}/L^{(i)}$ carries the invariant complex structure such that $G_0^{(i)}/(P_{x_i}^{(i)} \cap \tau P_{x_i}^{(i)})_0 \rightarrow G_0^{(i)}/L_0^{(i)}$ is holomorphic,

(ii) if the conditions (5.5) do not all hold then $L^{(i)} = G_0^{(i)}$.

We justify the terminology by observing that

(a) if $G_0^{(i)}/L^{(i)}$ is not a point, then it is an irreducible hermitian symmetric space of noncompact type, so $D(G_0, x)$ is a hermitian symmetric space of noncompact type (and possibly of dimension 0), and

(b) $D(G_0, x)$ is the “largest” hermitian symmetric space of non-compact type such that the natural map $G_0(x) \rightarrow D(G_0, x)$ is G_0 -equivariant and holomorphic.

5.7. THEOREM. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , and $x \in X$ such that $G_0(x)$ is open in X . Let*

$$\psi: G_0(x) \rightarrow D(G_0, x)$$

denote the natural projection onto the bounded symmetric domain subordinate to $G_0(x)$. Then the holomorphic functions on $G_0(x)$ are just the functions $\tilde{f} = f \cdot \psi$ where f is a holomorphic function on $D(G_0, x)$.

PROOF. We first consider the case where G_0 is simple. Let $\mathcal{K}_0 = \mathcal{K}_T + \mathcal{K}_T \subset \mathfrak{g}_0 \cap \mathfrak{o}_x$ be a maximally compact Cartan subalgebra of \mathfrak{g}_0 ; we have \mathfrak{o}_x^c specified by $\mathcal{K} = \mathcal{K}_0^c$. Let K be a maximal compact subgroup of G_0 such that $\mathcal{K}_T \subset \mathcal{K}$. If $\mathfrak{s} \subset \mathfrak{g}$, $z \in X$ and h is a smooth function on a neighborhood of z , then $\mathfrak{s}(h)_z$ denotes the set of values of derivatives of h at z by vector fields induced from elements of \mathfrak{s} .

Suppose that h is a nonconstant holomorphic function on $G_0(x)$. As $K(x)$ is a compact complex submanifold of $G_0(x)$ by Lemma 5.1, h must be constant on $K(x)$, so $\mathcal{K}(h)_x = 0$. The vector fields induced by $(\mathfrak{o}_x \cap \tau \mathfrak{o}_x)_0$ vanish at x , so also $(\mathfrak{o}_x \cap \tau \mathfrak{o}_x)_0(h)_x = 0$. If $(P_x \cap \tau P_x)_0 \not\subset K$ then \mathcal{K} and $(\mathfrak{o}_x \cap \tau \mathfrak{o}_x)_0$ generate \mathfrak{g}_0 , for \mathfrak{g}_0 is simple, so $\mathfrak{g}_0(h)_x = 0$; then the same argument shows $\mathfrak{g}_0(h)_z = 0$ for every $z \in G_0(x)$, so all first derivatives of h vanish, whence h is constant contrary to hypothesis. Thus $(P_x \cap \tau P_x)_0$ is compact and contained in K .

The compactness just proved shows $\mathcal{K}_0 = \mathcal{K}_T \subset \mathcal{K}$. Now the root system $\Delta = \Delta_K \cup \Delta_M$ where Δ_K is the root system of \mathcal{K}^c (compact roots) and Δ_M consists of the remaining roots (noncompact roots). If $\varphi \in \Delta$ then $\tau(\varphi) = -\varphi$ because \mathcal{K}_0 is compact. Thus $\tau\Phi^r = \Phi^r$ and $\tau\Phi^u = -\Phi^u$ where $\mathfrak{o}_x = \mathfrak{o}_\Phi$. In other words $\mathfrak{o}_x \cap \tau \mathfrak{o}_x = \mathfrak{o}_x^c$ and $\mathcal{K}^c + \mathfrak{o}_x = \mathcal{K}^c + \sum_{\Delta_M^+} \mathfrak{g}_\varphi$. Define $\mathfrak{q} = \mathcal{K}^c + \mathfrak{o}_x$. We have $\mathcal{K}^c(h)_x = 0$ because h is holomorphic and $\mathcal{K}(h)_x = 0$; we have $\mathfrak{o}_x(h)_x = 0$ because the vector fields induced by \mathfrak{o}_x vanish at x ; now $\mathfrak{q}(h)_x = 0$. If \mathfrak{q} generates \mathfrak{g} , then $\mathfrak{g}(h)_x = 0$, so $\mathfrak{g}_0(h)_x = 0$ and we contradict nonconstancy of h as above. Thus $\mathfrak{q} = \mathcal{K}^c + \sum_{\Delta_M^+} \mathfrak{g}_\varphi$ does not generate \mathfrak{g} . As G_0 is simple it follows that G_0/K is hermitian symmetric, \mathfrak{q} is a parabolic subalgebra of \mathfrak{g} containing \mathfrak{o}_x , and (Borel embedding) $gK \rightarrow gQ$ embeds G_0/K as an open G_0 -orbit on the complex flag manifold $Z = G/Q$. Let $\pi: X \rightarrow Z$ be the projection, $\pi(gP_x) = gQ$. Then π is holomorphic and its restriction to $G_0(x)$ is the projection $g(x) \rightarrow gK$ of $G_0(x)$ to G_0/K . Note that h is the lift of a function on G_0/K because it must be constant on the compact complex submanifolds $gK(x)$, $g \in G_0$, of $G_0(x)$.

Now we consider the general case, using the notation preceding Definition 5.6. Let h be a holomorphic function on $G_0(x)$. Let $1 \leq i \leq t$; fix $z_s \in G_0^{(s)}(x_s)$ for $s \neq i$, so $h(z_1, \dots, z_t) = h_i(z_i)$ is a holomorphic function on $G_0^{(i)}(x_i)$. We have just proved that h_i is constant if the conditions (5.5) do not all hold, and that h_i is the lift of a function on $G_0^{(i)}/K^{(i)}$ if the conditions (5.5) do all hold. This proves that $h = f \cdot \psi$ where $\psi: G_0(x) \rightarrow D(G_0, x)$ is the natural projection and f is a function $D(G_0, x)$. If f is not holomorphic, $\bar{\partial}f \neq 0$, so $\bar{\partial}h = \psi^* \bar{\partial}f \neq 0$ and h is not holomorphic; thus $h = f \cdot \psi$ where f is a holomorphic function on $D(G_0, x)$. Conversely, if f is a holomorphic function on $D(G_0, x)$, then its lift $f \cdot \psi$ is holomorphic because ψ is holomorphic. Q.E.D.

Two immediate consequences are:

5.8. COROLLARY. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , and $x \in X$ such that $G_0(x)$ is open in X .*

1. *The holomorphic functions separate points on $G_0(x)$, if and only if $(P_x \cap \tau P_x)_0$ is a maximal compact subgroup of G_0 and $G_0/(P_x \cap \tau P_x)_0$ is a hermitian symmetric coset space.*

2. *The following conditions are equivalent:*

(2a) *$G_0(x)$ carries a nonconstant holomorphic function.*

(2b) $\dim D(G_0, x) > 0$.

(2c) *Let $\mathfrak{K}_0 \subset \mathfrak{G}_0 \cap \mathfrak{O}_x$ be a maximally compact Cartan subalgebra of \mathfrak{G}_0 . Then \mathfrak{G}_0 has a simple ideal \mathfrak{G}'_0 , and \mathfrak{G}'_0 has a maximal compactly embedded subalgebra \mathfrak{K}' , such that*

(i) $\mathfrak{K}'_0 = \mathfrak{K}_0 \cap \mathfrak{K}'$ *is a Cartan subalgebra of \mathfrak{G}'_0 ,*

(ii) $\mathfrak{G}'_0 \text{ e} \cap \mathfrak{O}_x = \mathfrak{O}'_x$ *is given as $\mathfrak{O}'_{\Phi'}$ where $\Phi' \cup \{\varphi_1, \dots, \varphi_s\}$ is a system of simple $\mathfrak{K}'_0 \text{ e}$ -roots of $\mathfrak{G}'_0 \text{ e}$, and*

(iii) $\mathfrak{K}' \text{ e} = \mathfrak{O}'_{\Psi'}$ *where $\Psi' = \Phi' \cup \{\varphi_1, \dots, \varphi_{s-1}\}$ and φ_s has coefficient 1 in the highest root.*

Now we have seen that an extremely special situation is required in order that an open orbit $G_0(x) \subset X$ carry a nonconstant holomorphic function. In particular we cannot expect the closed G_0 -orbit to be the Bergman-Šilov boundary of an open orbit except in the usual [2] hermitian symmetric case.

6. Open orbits: invariant measure. In this section we develop the notion of measurable orbit for the case of an open orbit.

6.1. DEFINITION. *Let $X = G/P$ be a complex flag manifold, $x \in X$, and G_0 a real form of G . Then we say that $G_0(x)$ is a **measurable open orbit** if it is open in X and carries a G_0 -invariant volume element.*

The elementary tool required is

6.2. LEMMA. *Let $H_0 \subset L \subset G_0$ be connected Lie groups where G_0 is reductive and \mathfrak{H}_0 is a Cartan subalgebra of \mathfrak{G}_0 . Then G_0/L carries an invariant volume element if and only if L is reductive.*

PROOF. $\mathfrak{L}^e = \mathfrak{H} + \sum_{\Delta} \mathfrak{G}_{\varphi}$ and $\mathfrak{G} = \mathfrak{H} + \sum_{\Delta} \mathfrak{G}_{\varphi}$ where $\mathfrak{H} = \mathfrak{H}_0^e$, $\mathfrak{G} = \mathfrak{G}_0^e$ and the root systems $\Lambda \subset \Delta$. The representation χ of L on the tangent space of G_0/L complexifies to the restriction of $\text{ad}_{\mathfrak{G}}(L)$ to $\sum_{\Delta-\Lambda} \mathfrak{G}_{\varphi}$. Now G_0/L has an invariant volume element if and only if $\det \chi = 1$, and $\det \chi = 1$ if and only if $\text{ad}_{\mathfrak{G}}(\mathfrak{L})$ acts on $\sum_{\Delta-\Lambda} \mathfrak{G}_{\varphi}$ with trace 0. As $\text{ad}_{\mathfrak{G}}(\mathfrak{L})$ acts on \mathfrak{G} with trace 0, because \mathfrak{G} is reductive, now G_0/L carries a G_0 -invariant volume element if and only if $\text{ad}_{\mathfrak{L}}(\mathfrak{L})$ has trace 0, i.e. if and only if $\sum_{\Delta} \varphi = 0$, i.e. if and only if $\varphi \in \Lambda$ implies $-\varphi \in \Lambda$, i.e. if and only if L is reductive. Q.E.D.

Now we have a quick characterization

6.3. THEOREM. *Let $X = G/P$ be a complex flag manifold, $x \in X$, and G_0 a real form of G . Then the following conditions are equivalent, and each implies that $G_0(x)$ is open in X .*

- (1a) $G_0(x)$ is a measurable open orbit.
- (1b) $G_0(x)$ has a G_0 -invariant volume element.
- (1c) $G_0(x)$ has a G_0 -invariant, possibly indefinite, kaehler metric.
- (1d) The isotropy subgroup $G_0 \cap P_x$ is the centralizer in G_0 of a (compact) toral subgroup $T \subset G_0$.
- (2a) $\mathcal{O}_x \cap \tau \mathcal{O}_x$ is reductive, i.e. $\mathcal{O}_x \cap \tau \mathcal{O}_x = \mathcal{O}_x^r \cap \tau \mathcal{O}_x^r$.
- (2b) $\mathcal{O}_x \cap \tau \mathcal{O}_x = \mathcal{O}_x^r$.
- (2c) $\tau \mathcal{O}_x^u = \mathcal{O}_x^{-u}$, i.e. $\mathfrak{g} = \mathcal{O}_x + \tau \mathcal{O}_x^u$.
- (2d) $\mathcal{O}_x = \mathcal{O}_{\Phi}$ with $\tau \Phi^r = \Phi^r$ and $\tau \Phi^u = -\Phi^u$.

PROOF. Let $\mathfrak{H} \subset \mathfrak{P}_x$ be a τ -stable Cartan subalgebra of \mathfrak{G} and let Π be a simple root system such that $\mathcal{O}_x = \mathcal{O}_{\Phi}$ for a subset $\Phi \subset \Pi$. Lemma 2.10 says $(\mathcal{O}_x \cap \tau \mathcal{O}_x)^r = \mathcal{O}_x^r \cap \tau \mathcal{O}_x^r$, root system $\Phi^r \cap \tau \Phi^r$, and $(\mathcal{O}_x \cap \tau \mathcal{O}_x)^u = (\mathcal{O}_x^r \cap \tau \mathcal{O}_x^u) + (\mathcal{O}_x^u \cap \tau \mathcal{O}_x^r) + (\mathcal{O}_x^u \cap \tau \mathcal{O}_x^u)$, root system $(\Phi^r \cap \tau \Phi^u) \cup (\Phi^u \cap \tau \Phi^r) \cup (\Phi^u \cap \tau \Phi^u)$.

(2d) implies (2b) by summing root spaces and (2b) visibly implies (2a). If $\mathcal{O}_x \cap \tau \mathcal{O}_x$ is reductive then $\tau \Phi^u \cap (\Phi^r \cup \Phi^u)$ is empty, so $\tau \Phi^u = -\Phi^u$; thus (2a) implies (2c). If $\tau \Phi^u = -\Phi^u$ then τ preserves $\{\varphi \in \Delta: \varphi \notin \Phi^u \cup -\Phi^u\}$, which is Φ^r ; thus (2c) implies (2d). Now (2a), (2b), (2c) and (2d) are equivalent and each implies that $\Phi^u \cap \tau \Phi^u$ is empty, i.e. that $G_0(x)$ is open in X . It follows from Lemma 6.2 that (1a), (1b), (2a), (2b), (2c) and (2d) are equivalent.

Let $G_0(x)$ be a measurable open orbit, i.e. $\tau \Phi^r = \Phi^r$ and $\tau \Phi^u = -\Phi^u$. τ preserves \mathcal{O}_x^r , hence its center $\mathfrak{H}_{\Phi} = \{h \in \mathfrak{H}: \Phi(h) = 0\}$; thus $\mathfrak{G}_0 \cap \mathfrak{H}_{\Phi} = \mathfrak{H}_{\Phi}^r + \mathfrak{H}_{\Phi}^u$ sum of toral and vector parts. We define

$$A = \{ \varphi \in \Phi^u : \varphi(\mathcal{H}'_T) = 0 \} \quad \text{and} \quad \alpha = \sum_{\Phi^u} \varphi.$$

$\tau\Phi^u = -\Phi^u$ says $\tau(\alpha) = -\alpha$, so $\alpha \in i\mathcal{H}'_T$. If $\varphi \in \Phi^u$ then $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1(\mathcal{H}_\Phi) = 0$ and $\varphi_2 \in i\mathcal{H}'_T + \mathcal{H}'_V$. Projection to the trivial representation space of the Weyl group of \mathcal{O}^r_x on $i\mathcal{H}'_T + \mathcal{H}'_V$ annihilates φ_1 , and that Weyl groups preserves Φ^u , so $\alpha \in i\mathcal{H}'_T + \mathcal{H}'_V$. Now $\alpha \in i\mathcal{H}'_T$. If $\varphi \in \Phi^u$ then $\langle \varphi, \alpha \rangle > 0$. Thus A is empty. Now $\{ \varphi \in \Delta : \varphi(\mathcal{H}'_T) = 0 \}$, which contains Φ^r , contains no element of $\Phi^u \cup -\Phi^u$. In other words \mathcal{O}^r_x is the centralizer of \mathcal{H}'_T in \mathfrak{g} , so $G_0 \cap P_x = (P'_x)_0$ is the centralizer in G_0 of the (compact) torus $T = \exp(\mathcal{H}'_T)$. We have proved that (1a) implies (1d).

Let $G_0 \cap P_x$ be the centralizer of a torus $T \subset G_0$. Choose $t \in \mathfrak{J}$ such that $\mathfrak{g}_0 \cap \mathcal{O}_x$ is the centralizer of t in \mathfrak{g}_0 , and let η denote the left invariant 1-form on G_0 whose value at the identity is given by $\eta(h) = \langle t, h \rangle$. Then $\bar{\omega} = d\eta$ is a left invariant 2-form on G_0 , right invariant by $G_0 \cap P_x$, inducing a nondegenerate G_0 -invariant 2-form ω on $G_0(x) = G_0 / (G_0 \cap P_x)$. Now the invariant almost complex structure J on $G_0(x)$, together with ω , defines an invariant hermitian metric with kaehler form ω . As $d\bar{\omega} = d(d\eta) = 0$ is the lift of $d\omega$ to G_0 , it follows that $d\omega = 0$ and the metric is kaehler. Thus (1d) implies (1c). And obviously (1c) implies (1b) because a kaehler metric (of any signature) has a volume element. Q.E.D.

6.4. COROLLARY. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , and $x \in X$ such that $G_0(x)$ is open in X . If G_0 has its maximal compact subgroups K such that $\text{rank } K = \text{rank } G_0$, then $G_0(x)$ is a measurable open orbit.*

PROOF. Let $\mathcal{H} \subset \mathcal{O}_x$ be a τ -stable Cartan subalgebra of \mathfrak{g} , such that $\mathcal{H}_0 = \mathcal{H} \cap \mathfrak{g}_0$ is a maximally compact Cartan subalgebra of \mathfrak{g}_0 . If $\text{rank } K = \text{rank } G_0$, then \mathcal{H}_0 is compact, so $\tau\varphi = -\varphi$ for every \mathcal{H} -root φ ; then $\tau\Phi^r = \Phi^r$ and $\tau\Phi^u = -\Phi^u$ where $\mathcal{O}_x = \mathcal{O}_\Phi$. Q.E.D.

The condition $\text{rank } K = \text{rank } G_0$ is not necessary for an open orbit to be measurable. For example

6.5. COROLLARY. *Let $X = G/B$ be a complex flag manifold where B is a Borel subgroup of G . Let G_0 be a real form of G . Then every open G_0 -orbit on X is measurable.*

PROOF. $\mathfrak{B} = \mathcal{O}_\Phi$ where Φ is empty. Thus $\tau\Phi^r = \Phi^r$, empty set, and $\tau\Phi^u = \tau\Delta^+ = \Delta^- = -\Phi^u$. Q.E.D.

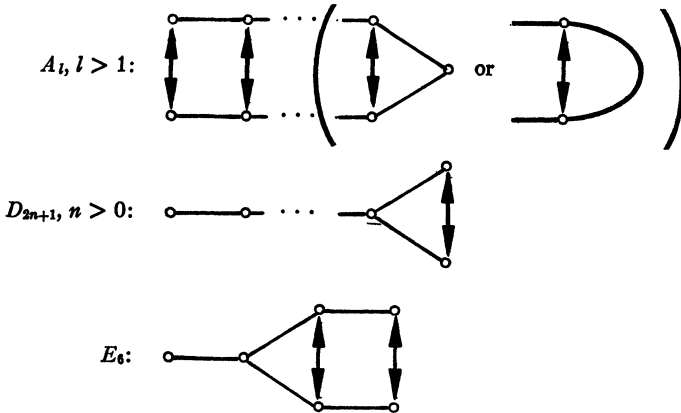
The main point of Corollaries 6.4 and 6.5 is that measurability does not depend on the choice of open orbit. In order to prove that in general we need the notion of opposed parabolic subgroups.

6.6. DEFINITIONS. The **opposition automorphism** of a reductive complex Lie algebra \mathfrak{g} relative to a Cartan subalgebra \mathfrak{H} is the automorphism

$$\nu: \mathfrak{g} \rightarrow \mathfrak{g} \text{ by } \nu(h) = -h \text{ for } h \in \mathfrak{H} \text{ and } \nu(e_\varphi) = e_{-\varphi}$$

where $\{h_\pi; e_\varphi\}$ is a Weyl basis of \mathfrak{g} . Note that any two opposition automorphisms are conjugate by an inner automorphism. Two subsets $\mathfrak{s}, \mathfrak{s}' \subset \mathfrak{g}$ are **opposed** if \mathfrak{s} is conjugate to $\nu(\mathfrak{s}')$ by an inner automorphism, for some (hence any) opposition automorphism of \mathfrak{g} .

Let \mathcal{P} be a parabolic subalgebra of a reductive or semisimple complex Lie algebra \mathfrak{g} . Choose a Cartan subalgebra $\mathfrak{H} \subset \mathcal{P}$ of \mathfrak{g} and a simple root system Π such that $\mathcal{P} = \mathcal{P}_\Phi$ for a subset $\Phi \subset \Pi$. Let ν be the opposition automorphism for this data and let w be the Weyl group element such that $w(\nu\Pi) = \Pi$. Then of course the parabolic $\nu(\mathcal{P})$ opposed to \mathcal{P} is conjugate to $\mathcal{P}_{w\nu\Phi}$. We can see the action of $w\nu$ on Π via the Dynkin diagram. It preserves each component, for ν preserves each simple ideal of \mathfrak{g} ; if a simple ideal has $-I$ in the Weyl group, i.e. if it is not of type A_l ($l > 1$), D_{2n+1} ($n > 0$) or E_6 , then $w = \nu = \nu^{-1}$ on that ideal, so $w\nu$ acts trivially on the corresponding component of the Dynkin diagram; if a simple ideal does not have $-I$ in the Weyl group, then $w\nu$ acts on its Dynkin diagram by:



6.7. THEOREM. Let $X = G/P$ be a complex flag manifold and G_0 a real form of G . Then the following conditions are equivalent:

- (i) \mathcal{P} and $\tau\mathcal{P}$ are opposed parabolic subalgebras of \mathfrak{g} .
- (ii) Some open G_0 -orbit on X is measurable.
- (iii) Every open G_0 -orbit on X is measurable.

PROOF. (iii) implies (ii) because G_0 has an open orbit.

Assume (ii) and let $G_0(x)$ be a measurable open orbit. $\mathfrak{K} \subset \mathcal{O}_x$ is a τ -stable Cartan subalgebra of \mathfrak{g} such that $\mathfrak{K}_0 = \mathfrak{K} \cap \mathfrak{g}_0$ is a maximally compact Cartan subalgebra of \mathfrak{g}_0 . Now $\mathcal{O}_x = \mathcal{O}_\Phi$ where Φ is contained in a simple root system Π . We take the opposition automorphism ν for \mathfrak{K} and Π . Using Theorem 6.3, $\tau\Phi^u = -\Phi^u = \nu\Phi^u$, so $\tau\mathcal{O}_x$ and $\nu\mathcal{O}_x$ each is the normalizer of \mathcal{O}_x^{-u} in \mathfrak{g} , whence $\tau\mathcal{O}_x = \nu\mathcal{O}_x$. Thus $\tau\mathcal{O}$ is conjugate to $\nu\mathcal{O}$, i.e. \mathcal{O} is opposed to $\tau\mathcal{O}$. That shows that (ii) implies (i).

Let \mathcal{O} and $\tau\mathcal{O}$ be opposed, let $G_0(x)$ be an open orbit, and let $\mathfrak{K}_0 \subset \mathfrak{g}_0 \cap \mathcal{O}_x$ be a maximally compact Cartan subalgebra of \mathfrak{g}_0 . Now $\mathcal{O}_x = \mathcal{O}_\Phi$ where Φ is contained in a simple root system Π such that $\tau(\Pi) = -\Pi$. $\tau\mathcal{O}_x$ and $\nu\mathcal{O}_x$ are conjugate parabolic subalgebras of \mathfrak{g} with a common Borel subalgebra $\mathfrak{K}_0^e + \sum_{\Delta^+} \mathfrak{g}_{-\varphi}$; thus $\tau\mathcal{O}_x = \nu\mathcal{O}_x$. Now $\mathcal{O}_x \cap \tau\mathcal{O}_x = \mathcal{O}_x \cap \nu\mathcal{O}_x = \mathcal{O}'_x$, so $G_0(x)$ is measurable by Theorem 6.3. Thus (i) implies (iii). Q.E.D.

We recover Corollary 6.4 from Theorem 6.7 because $\tau\varphi = -\varphi = \nu\varphi$ relative to a compact Cartan subalgebra $\mathfrak{K}_0 \subset \mathfrak{g}_0$, and we recover Corollary 6.5 because \mathfrak{B} , $\tau\mathfrak{B}$ and $\nu\mathfrak{B}$ are all conjugate. But, in addition, we have

6.8. COROLLARY. *Let $X = G/P$ be a complex flag manifold. Let G_0 be a real form of G such that \mathfrak{g}_0 is a direct sum of absolutely simple ideals $\mathfrak{g}_0^{(i)}$ each of which has one or more of the properties*

- (i) $\mathfrak{g}_0^{(i)}$ is not of type A_l ($l > 1$), D_{2n+1} ($n > 0$) nor E_6 ; or
- (ii) the maximal compactly embedded subalgebras of $\mathfrak{g}_0^{(i)}$ have maximal rank; or
- (iii) $\mathcal{O} \cap \mathfrak{g}^{(i)}$ is a Borel subalgebra of $\mathfrak{g}^{(i)} = \mathfrak{g}_0^{(i)e}$.

Then every open G_0 -orbit on X is measurable.

The key proviso of Corollary 6.8 is that the simple ideals of \mathfrak{g}_0 are absolutely simple, i.e. that τ preserves every simple ideal of \mathfrak{g} . But it is interesting to consider the other extreme.

Let G_0 be a complex semisimple connected Lie group viewed as a real Lie group. Then $G = G_0^e$ is $G_0 \times G_0$ and the embedding $G_0 \subset G$ is given by $g \rightarrow \delta(g) = (g, g)$. τ acts on G by $\tau(g_1, g_2) = (g_2, g_1)$. $P = P_1 \times P_2$ where P_i are parabolic subgroups of G_0 . In particular $\tau(P) = P_2 \times P_1$ where $P = P_1 \times P_2$. The opposition automorphism acts by $\nu(P) = \nu(P_1) \times \nu(P_2)$ where $P = P_1 \times P_2$. Thus \mathcal{O} and $\tau\mathcal{O}$ are opposed if and only if \mathcal{O}_1 and \mathcal{O}_2 are opposed. In other words, the open G_0 -orbits on G/P are measurable if and only if $P = P_1 \times P_2$, \mathcal{O}_1 and \mathcal{O}_2 opposed, i.e. if and only if $P = P_1 \times \nu P_1$. If G_0 has rank r as a complex group, then G has rank $2r$, so G has 2^{2r} conjugacy classes of parabolic subgroups $P = P_1 \times P_2$ and just 2^r of them are of the form $P_1 \times \nu P_1$. Thus "most" parabolic sub-

groups $P \subset G$ have the property that the open G_0 -orbits on G/P are not measurable.

7. Integrable orbits. Let $X = G/P$ be a complex flag manifold, $x \in X$ and G_0 a real form of G . We observe $\mathcal{P}_x + \tau\mathcal{P}_x = \mathfrak{g} \Leftrightarrow \mathcal{P}_x^{-u} \subset \tau\mathcal{P} \Leftrightarrow \mathcal{P}_x^{-u} \cap \tau\mathcal{P}_x^{-u} = 0 \Leftrightarrow \mathcal{P}_x^u \cap \tau\mathcal{P}_x^u = 0 \Leftrightarrow G_0(x)$ is open in X . This leads us to some definitions whose mystery will evaporate in §9.

7.1. DEFINITIONS. Let $X = G/P$ be a complex flag manifold, $x \in X$ and G_0 a real form of G . If $\mathcal{P}_x + \tau\mathcal{P}_x$ is a subalgebra of \mathfrak{g} , then we say that \mathcal{P}_x is an **integrable parabolic subalgebra** of \mathfrak{g} , P_x is an **integrable parabolic subgroup** of G , x is an **integrable point** of X , and $G_0(x)$ is an **integrable G_0 -orbit** on X .

These notions are, of course, dependent on choice of the real form G_0 . The basic facts are given by

7.2. THEOREM. Let \mathfrak{g} be a complex semisimple Lie algebra, \mathcal{P} a parabolic subalgebra, and \mathfrak{g}_0 a real form of \mathfrak{g} . Denote

$$\mathfrak{u} = \mathcal{P}^u \cap \tau\mathcal{P}^u \quad \text{and} \quad \mathfrak{N} \text{ is the normalizer of } \mathfrak{u} \text{ in } \mathfrak{g}.$$

Then the following conditions are equivalent:

- (i) \mathcal{P} is integrable, i.e. $\mathcal{P} + \tau\mathcal{P}$ is a subalgebra of \mathfrak{g} .
- (ii) $\mathcal{P} + \tau\mathcal{P} = \mathfrak{N}$.
- (iii) $\mathcal{P} + \tau\mathcal{P}$ is an algebra and \mathfrak{u} is its nilpotent radical.
- (iv) \mathfrak{u} is the nilpotent radical of \mathfrak{N} .
- (v) \mathcal{P} normalizes \mathfrak{u} , i.e. $\mathcal{P} \subset \mathfrak{N}$.

PROOF. Relative to a τ -stable Cartan subalgebra $\mathfrak{H} \subset \mathcal{P}$ of \mathfrak{g} and a subset Φ of a simple root system such that $\mathcal{P} = \mathcal{P}_\Phi$, we have $\mathcal{P} + \tau\mathcal{P} = \mathfrak{H} + \sum_{\Gamma} \mathfrak{g}_\Gamma$ where $\Gamma = \Phi^r \cup \tau\Phi^r \cup \Phi^u \cup \tau\Phi^u$. If $\varphi \in \Gamma$ and $-\varphi \notin \Gamma$ it follows that $\varphi \in \Phi^u \cup \tau\Phi^u$, $\varphi \notin \Phi^r \cup \tau\Phi^r$ and $\varphi \notin \Phi^r \cap \tau\Phi^r$; then $\varphi \in \Phi^u \cap \tau\Phi^u$ or $\varphi \in \Phi^u \cap -\tau\Phi^u$ or $\varphi \in -\Phi^u \cap \tau\Phi^u$; if $\varphi > 0$ and $\tau\varphi > 0$ we conclude that $\varphi \in \Phi^u \cap \tau\Phi^u$.

Let $\mathcal{P} + \tau\mathcal{P}$ be an algebra. The discussion above shows that it has nilpotent radical \mathfrak{u} . Thus (i) implies (iii). A parabolic subalgebra is the normalizer of its nilpotent radical; thus (iii) implies (ii). And visibly (ii) implies (i). Now (i), (ii) and (iii) are equivalent.

If $\mathcal{P} + \tau\mathcal{P} = \mathfrak{N}$ then $\mathcal{P} \subset \mathfrak{N}$; thus (ii) implies (v). If $\mathcal{P} \subset \mathfrak{N}$ then \mathfrak{N} is parabolic; $\tau\mathfrak{u} = \mathfrak{u}$ implies $\tau\mathfrak{N} = \mathfrak{N}$ so $\mathcal{P} + \tau\mathcal{P} \subset \mathfrak{N}$; now the discussion of the first paragraph of the proof shows that \mathfrak{u} contains the nilpotent radical of \mathfrak{N} ; thus (v) implies (iv). Now we need only prove that (iv) implies (ii).

Let \mathfrak{u} be the nilpotent radical of \mathfrak{N} . If \mathfrak{N} is a parabolic subalgebra of \mathfrak{g} , then $\mathfrak{N} = \mathcal{P}_\Psi$ with $\Phi \subset \Psi$ such that $\Psi^u = \Phi^u \cap \tau\Phi^u$. As Δ^+ is the dis-

joint union of $\Phi^r \cap \Delta^+$, $\Phi^u \cap (\tau\Phi^r \cup -\tau\Phi^u)$ and $\Phi^u \cap \tau\Phi^u$, and also is the disjoint union of $\Psi^r \cap \Delta^+$ and $\Psi^u = \Phi^u \cap \tau\Phi^u$, it follows that $\Psi^r \cap \Delta^+ = (\Phi^r \cap \Delta^+) \cup (\Phi^u \cap \tau\Phi^r) \cup (\Phi^u \cap -\tau\Phi^u)$ so $\mathfrak{N} = \mathcal{P}^r + \mathcal{P}^u + \tau\mathcal{P}^r + \tau\mathcal{P}^u = \mathcal{P} + \tau\mathcal{P}$. Thus (iv) implies (ii) provided that (iv) forces \mathfrak{N} to be parabolic. In other words, the theorem is reduced to

7.3. LEMMA. *Let \mathfrak{G} be a complex semisimple Lie algebra, let \mathfrak{u} be a nilpotent⁵ subalgebra, and let \mathfrak{N} be the normalizer of \mathfrak{u} in \mathfrak{G} . Suppose that \mathfrak{N} contains a Cartan subalgebra \mathfrak{H} of \mathfrak{G} and that \mathfrak{u} is the nilpotent⁵ radical of \mathfrak{N} . Then \mathfrak{N} is a parabolic subalgebra of \mathfrak{G} .*

PROOF. $\mathfrak{N} = \mathfrak{L} + \mathfrak{u}$ semidirect where $\mathfrak{H} \subset \mathfrak{L}$ and \mathfrak{L} is reductive. $\mathfrak{u} = \sum_A \mathfrak{G}_\varphi$ for some set A of \mathfrak{H} -roots of \mathfrak{G} . $\mathfrak{L} = \mathfrak{L}' \oplus \mathfrak{s}$ where $\mathfrak{s} \subset \mathfrak{H}$ is the center of \mathfrak{L} and $\mathfrak{L}' = [\mathfrak{L}, \mathfrak{L}]$ is the semisimple part. $\mathfrak{H} + \mathfrak{u}$ is contained in a Borel subalgebra of \mathfrak{G} , so we order the roots such that $\varphi > 0$ for every $\varphi \in A$.

Let \mathfrak{L} be semisimple, let W_L be its Weyl group and define $\alpha = \sum_A \varphi$. Then α is a W_L -invariant positive sum of positive roots. The action of W_L on \mathfrak{H} has no trivial subrepresentation. Thus $\alpha = 0$ and A is empty. It follows that $\mathfrak{u} = 0$, so $\mathfrak{N} = \mathfrak{G}$, which is a parabolic subalgebra of \mathfrak{G} .

Now we may assume \mathfrak{L} not semisimple, so its center $\mathfrak{s} \neq 0$. Define $B = \{\varphi \in A : \varphi(\mathfrak{s}) = 0\}$ and $\beta = \sum_B \varphi$. As above it follows that $\beta = 0$ so B is empty. Let Z be the centralizer of \mathfrak{s} in \mathfrak{G} and order the weights of \mathfrak{s} so that positive weight spaces are sums of positive root spaces. Then the corresponding parabolic subalgebra \mathcal{Q} of \mathfrak{G} has $Z = \mathcal{Q}^r$, and $\mathfrak{u} \subset \mathcal{Q}^u$ because $Z \cap \mathfrak{u} = 0$. Let $\mathfrak{W} = \mathfrak{N} \cap \mathcal{Q}^u$; so $\mathfrak{u} \subset \mathfrak{W} \subset \mathcal{Q}^u$. If $\mathfrak{u} \neq \mathcal{Q}^u$ then $\mathfrak{u} \neq \mathfrak{W}$ because a proper nilpotent subalgebra of a nilpotent Lie algebra cannot be its own normalizer. But $\mathfrak{N} = \mathfrak{L} + \mathfrak{u}$ with $\mathfrak{L} \subset \mathcal{Q}^r$ and \mathfrak{u} is the nilpotent radical of \mathfrak{N} ; thus $\mathfrak{u} = \mathfrak{N} \cap \mathcal{Q}^u = \mathfrak{W}$; now $\mathfrak{u} = \mathcal{Q}^u$ so $\mathfrak{N} = \mathcal{Q}$. Q.E.D.

Now we go on to construct all integrable G_0 -orbits on $X = G/P$.

Let \mathfrak{G} be a complex semisimple Lie algebra, \mathfrak{H} a Cartan subalgebra and Π a simple root system. We know that each conjugacy class of parabolic subalgebras of \mathfrak{G} is represented by just one algebra \mathcal{P}_Φ , $\Phi \subset \Pi$; in particular, if \mathcal{P} and \mathcal{Q} are parabolic subalgebras of \mathfrak{G} , then the conditions

- (i) \mathcal{Q} contains a conjugate of \mathcal{P} ,
 - (ii) \mathcal{P} is contained in a conjugate of \mathcal{Q} , and
 - (iii) \mathcal{P} is conjugate to \mathcal{P}_Φ , \mathcal{Q} to \mathcal{P}_Ψ , with $\Phi \subset \Psi \subset \Pi$,
- are equivalent.

⁵ This means that \mathfrak{u} is represented by nilpotent matrices in the adjoint representation of \mathfrak{G} .

7.4. DEFINITION. We write $\mathcal{P} \prec \mathcal{Q}$ to denote the equivalent conditions (i), (ii) and (iii) above.

Now let \mathfrak{g}_0 be a real form of \mathfrak{g} and assume \mathcal{H} stable under the complex conjugation τ of \mathfrak{g} over \mathfrak{g}_0 . Let $w \in W_{\mathcal{G}}$ be the Weyl group element which carries the simple root system $\tau(\Pi)$ back to Π . We have a $(w\tau)$ -stable subset $\Psi_0 \subset \Pi$ determined by: $(\mathfrak{M} + \mathcal{A} + \mathfrak{N})^{\circ}$ is conjugate to \mathcal{P}_{Ψ_0} where $\mathfrak{M} + \mathcal{A} + \mathfrak{N}$ is the minimal parabolic subalgebra of \mathfrak{g}_0 . Now the conjugacy classes of parabolic subalgebras, such that the class contains an algebra defined/R (i.e. τ -stable), are represented by the \mathcal{P}_{Ψ} such that

$$(7.5) \quad (a) (w\tau)\Psi = \Psi \quad \text{and} \quad (b) \Psi_0 \subset \Psi \subset \Pi.$$

In order to apply the criterion (7.5), one uses the fact [15, p. 909] that, if \mathfrak{g} is simple, then $w\tau$ acts trivially on Π except in the cases:

\mathfrak{g}_0	action of $w\tau$ on Π
$\mathfrak{SU}^*(n)$	
$\mathfrak{SO}^*(4n+2)$ $\mathfrak{SO}^{2k}(4n+2)$ $\mathfrak{SO}^{2k+1}(4n)$	
$\mathfrak{E}_6, \mathfrak{F}_4; \mathfrak{E}_6, \mathfrak{C}_4$	

Now suppose that we have a fixed parabolic subalgebra $\mathcal{P} \subset \mathfrak{g}$. We look for the τ -stable parabolic subalgebras $\mathcal{Q} \subset \mathfrak{g}$ such that $\mathcal{P} \prec \mathcal{Q}$. Given such an algebra \mathcal{Q} we have a conjugate \mathcal{P}_x of \mathcal{P} such that $\mathcal{P}_x + \tau\mathcal{P}_x \subset \mathcal{Q}$. Suppose that \mathcal{P} is conjugate to \mathcal{P}_{Φ} and \mathcal{Q} is conjugate to \mathcal{P}_{Ψ} ; then $\Phi \subset \Psi$ and Ψ satisfies (7.5). In other words

$$(7.6) \quad (a) (w\tau)\Psi = \Psi \quad \text{and} \quad (b) \{\Psi_0 \cup \Phi \cup (w\tau)\Phi\} \subset \Psi \subset \Pi.$$

Conversely, if Ψ satisfies (7.6), then \mathcal{O}_Ψ has a τ -stable conjugate \mathcal{Q} because Ψ satisfies (7.5), and $\mathcal{O} \prec \mathcal{Q}$ because $\Phi \subset \Psi$. This proves the first statement of

7.7. THEOREM. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , \mathcal{H} a τ -stable Cartan subalgebra of \mathfrak{g} , Π a simple \mathcal{H} -root system, and $\Phi \subset \Pi$ such that \mathcal{O} is conjugate to \mathcal{O}_Φ . Let $w \in W_G$ such that $(w\tau)\Pi = \Pi$ and define*

$$(7.8) \quad \Psi_X = \Psi_0 \cup \Phi \cup (w\tau)\Phi.$$

1. *The \mathfrak{g} -conjugacy classes of parabolic subalgebras $\mathcal{Q} \subset \mathfrak{g}$ such that*

$$\mathcal{O} \prec \mathcal{Q} \quad \text{and} \quad \tau(\mathcal{Q}) = \mathcal{Q}$$

are the classes represented by algebras \mathcal{O}_Ψ such that

$$(w\tau)\Psi = \Psi \quad \text{and} \quad \Psi_X \subset \Psi \subset \Pi.$$

2. *If $\mathcal{Q} \subset \mathfrak{g}$ is a τ -stable parabolic subalgebra then the following conditions are equivalent:*

(2a) $\mathcal{O} \prec \mathcal{Q}$.

(2b) $\mathcal{Q} = \mathcal{O}_x + \tau\mathcal{O}_x$ for some (necessarily integrable) $x \in X$.

(2c) \mathcal{Q} is conjugate to \mathcal{O}_Ψ with $\Psi_X \subset \Psi \subset \Pi$.

PROOF. Statement (1) was proved in the discussion preceding the theorem.

If $\tau(\mathcal{Q}) = \mathcal{Q}$ and $\mathcal{O} \prec \mathcal{Q}$ then (1) implies that \mathcal{Q} is conjugate to \mathcal{O}_Ψ with $\Psi_X \subset \Psi \subset \Pi$. If $\tau(\mathcal{Q}) = \mathcal{Q}$ and \mathcal{Q} is conjugate to \mathcal{O}_Ψ with $\Psi_X \subset \Psi \subset \Pi$ then $\Phi \subset \Psi$ so $\mathcal{O} \prec \mathcal{Q}$. Now (2a) and (2c) are equivalent. As (2b) trivially implies (2a), we now need only check that (2a) implies (2b).

Let $\tau(\mathcal{Q}) = \mathcal{Q}$ and $\mathcal{O} \prec \mathcal{Q}$. If we have a Borel subalgebra $\mathcal{B} \subset \mathcal{Q}$ of \mathfrak{g} such that $\mathcal{B} + \tau\mathcal{B} = \mathcal{Q}$, then we choose a conjugate \mathcal{O}_x of \mathcal{O} such that $\mathcal{B} \subset \mathcal{O}_x \subset \mathcal{Q}$; that gives $\mathcal{Q} = \mathcal{B} + \tau\mathcal{B} \subset \mathcal{O}_x + \tau\mathcal{O}_x \subset \mathcal{Q} + \tau\mathcal{Q} = \mathcal{Q}$, and we conclude $\mathcal{O}_x + \tau\mathcal{O}_x = \mathcal{Q}$. Now the proof of Theorem 7.7 comes down to the proof of

7.9 LEMMA. *Let \mathfrak{g} be a complex semisimple Lie algebra, \mathfrak{g}_0 a real form of \mathfrak{g} , and \mathcal{Q} a τ -stable parabolic subalgebra of \mathfrak{g} . Then there is a Borel subalgebra $\mathcal{B} \subset \mathcal{Q}$ of \mathfrak{g} with $\mathcal{B} + \tau\mathcal{B} = \mathcal{Q}$.*

PROOF. $\mathcal{Q} = \mathcal{Q}^r + \mathcal{Q}^u$ with each summand stable under τ . Now $\mathcal{Q}_0^r = \mathfrak{g}_0 \cap \mathcal{Q}^r$ is a real form of \mathcal{Q}^r and is a reductive subalgebra of maximal rank in \mathfrak{g}_0 . Let $\mathcal{H}_0 = \mathcal{H}_T + \mathcal{H}_T$ be a maximally compact Cartan subalgebra of \mathcal{Q}_0^r . Apply Theorem 4.5 to the action of \mathcal{Q}_0^r on $\mathcal{Q}^r / (\text{Borel subgroup})$; it shows that \mathcal{Q}^r has a Borel subalgebra \mathcal{B}_1 such that $\mathcal{B}_1 + \tau\mathcal{B}_1 = \mathcal{Q}^r$.

Now $\mathfrak{B} = \mathfrak{B}_1 + \mathcal{Q}^u$ is a Borel subalgebra of \mathfrak{g} , and $\mathfrak{B} + \tau\mathfrak{B} = \mathcal{Q}^r + \mathcal{Q}^u = \mathcal{Q}$.
 Q.E.D.

We extend the proof that (2a) implies (2b) in Theorem 7.7 to an existence and enumeration theorem for integrable orbits.

7.10. THEOREM. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , and \mathfrak{K} a τ -stable Cartan subalgebra of \mathfrak{g} . Let Π be a simple \mathfrak{K} -root system, $w \in W_G$ such that $(wr)\Pi = \Pi$, $\Phi \subset \Pi$ such that \mathcal{O} is conjugate to \mathcal{O}_Φ , and $\Psi_X = \Phi \cup (wr)\Phi \cup \Psi_0$, (wr) -stable subset of Π . For every set Ψ such that $\Psi_X \subset \Psi \subset \Pi$ and $(wr)\Psi = \Psi$, choose a τ -stable parabolic subalgebra $\mathcal{Q}_\Psi \subset \mathfrak{g}$ which is conjugate to \mathcal{O}_Ψ , and choose a maximally compact Cartan subalgebra $\mathfrak{K}_{\Psi,0}$ of the real form $\mathfrak{g}_0 \cap \mathcal{Q}_\Psi^r$ of \mathcal{Q}_Ψ^r .*

1. *If \mathfrak{B}_1 is a Borel subalgebra of \mathcal{Q}_Ψ^r which contains $\mathfrak{K}_\Psi = \mathfrak{K}_{\Psi,0}$ and such that $\mathfrak{B}_1 + \tau\mathfrak{B}_1 = \mathcal{Q}_\Psi^u$, and if \mathfrak{B} denotes the Borel subalgebra $\mathfrak{B}_1 + \mathcal{Q}_\Psi^u$ of \mathfrak{g} , and if \mathcal{O}_z is the conjugate of \mathcal{O} such that $\mathfrak{B} \subset \mathcal{O}_z \subset \mathcal{Q}_\Psi$, then*

- (1a) x is an integrable point of X ,
- (1b) $G_0(x)$ is an integrable G_0 -orbit on X , and
- (1c) $\mathcal{O}_z + \tau\mathcal{O}_z = \mathcal{Q}_\Psi$.

2. *If $z \in X$ is an integrable point, and if the parabolic subalgebra $\mathcal{O}_z + \tau\mathcal{O}_z$ is conjugate to \mathcal{O}_Ψ , then in (1) we can choose the Borel subalgebra \mathfrak{B}_1 of \mathcal{Q}_Ψ^r in such a manner that $z \in G_0(x)$.*

3. *For each set Ψ , $\Psi_X \subset \Psi \subset \Pi$ and $(wr)\Psi = \Psi$, choose one point $x_\Psi \in X$ as constructed in (1). Let $\mathcal{Q}_{\Psi,0}^r$ denote the identity component of $G_0 \cap \mathcal{Q}_\Psi^r$. Then $Q_\Psi(x_\Psi) = \mathcal{Q}_\Psi^r(x_\Psi) = \mathcal{Q}_\Psi^r / (\mathcal{Q}_\Psi^r \cap P_{x_\Psi}^r)$ is a complex flag manifold X_Ψ , and the open $\mathcal{Q}_{\Psi,0}^r$ -orbits on X_Ψ are in one to one correspondence with the integrable orbits $G_0(x)$ such that $\mathcal{O}_z + \tau\mathcal{O}_z$ is conjugate to \mathcal{O}_Ψ .*

4. *Let K_Ψ be a maximal compact subgroup of $\mathcal{Q}_{\Psi,0}^r$ such that \mathfrak{K}_Ψ contains the compact part of $\mathfrak{K}_{\Psi,0}$ and is orthogonal to the noncompact part. Then the integrable orbits $G_0(x) \subset X$ such that $\mathcal{O}_z + \tau\mathcal{O}_z$ is conjugate to \mathcal{O}_Ψ are enumerated by the double coset space*

$$W_{K_\Psi} \backslash W_{\mathcal{Q}_\Psi^r}^{\mathfrak{K}_{\Psi,0}} / W_{P_{x_\Psi}^r \cap \tau P_{x_\Psi}^r}^{\mathfrak{K}_{\Psi,0}}$$

where superscript $\mathfrak{K}_{\Psi,0}$ means the subgroup of the Weyl group which preserves $\mathfrak{K}_{\Psi,0}$ in \mathfrak{K}_Ψ .

PROOF. Statement (1) is the construction employed in proving that (2a) implies (2b) in Theorem 7.7.

Let $\mathcal{O}_z + \tau\mathcal{O}_z$ be conjugate to \mathcal{O}_Ψ . Then \mathcal{Q}_Ψ and $\mathcal{O}_z + \tau\mathcal{O}_z$ are parabolic subalgebras of \mathfrak{g} which are conjugate by an element of G and are defined/R; so they are conjugate by an element of G_0 . Replacing z by $g(z)$ for some $g \in G_0$ we may assume $\mathcal{O}_z + \tau\mathcal{O}_z = \mathcal{Q}_\Psi$. Further replacing z

by $g(z)$ for some $g \in G_0$ we may also assume $\mathcal{K}_{\Psi,0} \subset \mathcal{P}_z$. Let \mathcal{B}_2 be a Borel subalgebra of $\mathcal{P}'_z \cap \tau \mathcal{P}'_z$ containing \mathcal{K}_{Ψ} such that $\mathcal{B}_2 + \tau \mathcal{B}_2 = \mathcal{P}'_z \cap \tau \mathcal{P}'_z$; such \mathcal{B}_2 exist because $\mathcal{K}_{\Psi,0}$ is maximally compact in $\mathcal{G}_0 \cap \mathcal{Q}'_{\Psi}$, hence in $\mathcal{G}_0 \cap (\mathcal{P}'_z \cap \tau \mathcal{P}'_z)$. Now $\mathcal{B} = \mathcal{B}_2 + (\mathcal{P}'_z \cap \tau \mathcal{P}_z^{-u}) + \mathcal{P}_z^u$ is a Borel subalgebra of \mathcal{G} such that $\mathcal{B} \subset \mathcal{P}_z \subset \mathcal{Q}_{\Psi}$ and $\mathcal{B} + \tau \mathcal{B} \supset \mathcal{P}_z$. As $\mathcal{P}_z + \tau \mathcal{P}_z = \mathcal{Q}_{\Psi}$ now $\mathcal{B} + \tau \mathcal{B} = \mathcal{Q}_{\Psi}$. Decompose $\mathcal{B} = \mathcal{B}_1 + \mathcal{Q}'_{\Psi}$ where \mathcal{B}_1 is a Borel subalgebra of \mathcal{Q}'_{Ψ} containing \mathcal{K}_{Ψ} ; now $\mathcal{B}_1 + \tau \mathcal{B}_1 = \mathcal{Q}'_{\Psi}$. Thus, having replaced z by $g(z)$ for some $g \in G_0$, we end up with z as one of the points $x \in X$ constructed in (1). In other words, we started with $z \in G_0(x)$. Statement (2) is proved.

The choice available in the construction of statement (1) is the choice of \mathcal{B}_1 . That choice is free, without changing $G_0(x)$, \mathcal{Q}_{Ψ} or $\mathcal{K}_{\Psi,0}$, up to conjugation by an element of $\mathcal{Q}_{\Psi,0}$ which normalizes $\mathcal{K}_{\Psi,0}$, i.e. up to conjugation by an element of $G_0 \cap \mathcal{Q}'_{\Psi}$, i.e. up to conjugation by an element of $\mathcal{Q}'_{\Psi,0}$. This proves (3), and (4) follows from Theorem 4.9. Q.E.D.

Theorems 7.7 and 7.10 give a complete description of the integrable G_0 -orbits on X in terms of the isotropy subalgebras of \mathcal{G} at integrable points. But they do not "locate" the integrable orbits on X .

The complex flag manifold $X = G/P$ always has some integrable G_0 -orbits, for at the beginning of this §7 we observed that $\mathcal{P}_z + \tau \mathcal{P}_z = \mathcal{G}$ if and only if $G_0(x)$ is open in X . But it can happen that the open orbits are the only integrable G_0 -orbits:

7.11. PROPOSITION. *Let $X = G/P$ be a complex flag manifold where P is a maximal parabolic subgroup⁶ of G . Let $x \in X$ and let G_0 be a real form of G . Then the orbit $G_0(x)$ is integrable if and only if*

- (a) $G_0(x)$ is open in X , or
- (b) $G_0(x)$ is the closed orbit and $\dim_{\mathbb{R}} G_0(x) = \dim_{\mathbb{C}} X$.

PROOF. Let $G_0(x)$ be integrable. If $\tau \mathcal{P}_z = \mathcal{P}_z$ we are in case (b). If $\tau \mathcal{P}_z \neq \mathcal{P}_z$ then $\mathcal{P}_z + \tau \mathcal{P}_z = \mathcal{G}$ by maximality of P , so we are in case (a). Q.E.D.

7.12. COROLLARY. *Let $X_0 = G_0/K$ be a hermitian symmetric space of noncompact type, $G = G_0^{\mathbb{C}}$, and $X = G_u/K = G/P$ the compact dual of X . Decompose X_0 as a product of irreducible factors $X_0^{(i)} = G_0^{(i)}/K^{(i)}$, G as a product of simple groups $G^{(i)} = G_0^{(i)\mathbb{C}}$, and X as a product of irreducible factors $X^{(i)} = G_u^{(i)}/K^{(i)} = G^{(i)}/P^{(i)}$, so that the G_0 -orbits on X are just the products of $G_0^{(i)}$ -orbits on the $X^{(i)}$. Let $x = (x_1, \dots, x_t) \in X^{(1)} \times \dots$*

⁶ This means that $P \neq G$ and there is no parabolic subgroup $Q \subset G$ with $P \subsetneq Q \subsetneq G$. In other words, if $\mathcal{P} = \mathcal{P}_{\Phi}$ where Φ is in a simple root system Π , then $|\Pi| = |\Phi| + 1$.

$\times X^{(i)} = X$. Then the orbit $G_0(x) \subset X$ is integrable if, and only if, for each i ,

either $G_0^{(i)}(x_i)$ is open in $X^{(i)}$
 or $X_0^{(i)}$ is of tube type and $G_0^{(i)}(x_i)$ is its Bergman-Šilov boundary in $X^{(i)}$.

PROOF. We may assume X_0 irreducible. Then $t = 1$ and P is a maximal parabolic subgroup of G , and the two cases of Corollary 7.12 correspond to the respective cases (a) and (b) of Proposition 7.11.

Q.E.D.

CHAPTER II. DECOMPOSITION OF A REAL GROUP ORBIT INTO COMPLEX ANALYTIC PIECES

Let $X = G/P$ be a complex flag manifold and let G_0 be a real form of G . §8 and §9 are an analysis of the “amount” of complex structure inherited from X by an arbitrary orbit $G_0(x) \subset X$.

In §8 the orbit $G_0(x)$ is decomposed into “holomorphic arc components” $S_{[z]}$. They are equivalence classes for the G_0 -invariant relation: $y \in S_{[z]}$ if and only if it can be joined to z by a chain of holomorphic (in X) arcs in $G_0(x)$. The holomorphic arc components are real analytic submanifolds of $G_0(x)$. In fact,

$$N_{[z],0} = \text{identity component of } \{g \in G_0: gS_{[z]} = S_{[z]}\}$$

is a closed analytic subgroup of G_0 that is transitive on $S_{[z]}$. We approach analysis of the holomorphic arc components by studying the algebra $\mathfrak{N}_{[z]} = \mathfrak{N}_{[z],0}^e$.

Theorem 8.5, Proposition 8.7 and Theorem 8.9 determine subspaces $Q_{[z]} \subset \mathfrak{M}_{[z]} \subset \{\mathfrak{N}_{[z]} \cap (\mathcal{O}_z + \tau\mathcal{O}_z)\}$ stable under τ , such that $Q_{[z]}$ is a parabolic subalgebra of \mathfrak{g} (so $N_{[z],0}$ is the identity component of a parabolic subgroup of G_0), $S_{[z]}$ is a complex submanifold of X if and only if $\mathfrak{M}_{[z]} = \mathfrak{N}_{[z]}$, and $\mathfrak{M}_{[z]} = \mathfrak{N}_{[z]}$ if and only if $\mathfrak{M}_{[z]}$ is an algebra. Thus one can decide whether the holomorphic arc components of $G_0(x)$ are complex submanifolds of X , and in that case their structure is determined in principle. If they are not complex then their submanifolds $Q_{[z],0}(z) \subset S_{[z]}$, where $Q_{[z],0}$ is the analytic group for $Q_{[z]} \cap \mathfrak{g}_0$, are acceptable substitutes which are complex submanifolds of X ; we call them the “algebraic arc components” of $G_0(x)$.

If $P_1 \subset P_2$ are parabolic subgroups of G , $X_i = G/P_i$ and $\pi: X_1 \rightarrow X_2$ the projection, $x_i \in X_i$ with $\pi(x_1) = x_2$, then Proposition 8.14 relates the holomorphic and algebraic arc component structure of the $G_0(x_i)$.

§8 ends with a description of the space of holomorphic arc components of $G_0(x)$.

In §9 we study certain global conditions on the holomorphic arc components of $G_0(x)$. They are

- (i) partially complex: $S_{[x]}$ is a complex manifold,
- (ii) flag type: $N_{[x]}(x)$ is a complex flag manifold,
- (iii) measurable: $S_{[x]}$ has an $N_{[x],0}$ -invariant measure,
- (iv) polarized: we can arrange \mathcal{H} so that $\tau\mathcal{O}_x^r = \mathcal{O}_x^r$,
- (v) integrable: $\mathcal{O}_x + \tau\mathcal{O}_x$ is an algebra.

The important ones are "partially complex" and "measurable," and the others are useful in their study. Theorem 9.2 gives the basic analysis of measurable orbits, describing $\mathfrak{N}_{[x]}$ and proving them partially complex and of flag type. Theorem 9.7 performs a similar task for orbits of flag type. Then Theorem 9.9 shows that, if $G_0(x)$ is polarized, then the conditions "measurable," "integrable," and "partially complex and of flag type" coincide. Various corollaries complete the general part of §9.

The specific part of §9 starts with Theorem 9.12, which says that the closed G_0 -orbit on X is automatically measurable. That result is related [12] to the existence of principal series representations of G_0 . Proposition 9.18 and Corollary 9.21 extend the method of Theorem 9.12.

§9 ends with a description of the case where X is a hermitian symmetric space and G_0 is an *arbitrary* real form of G . Proposition 9.24 gives measurability criteria for the open G_0 -orbits, and one criterion is that they be indefinite-kaehler symmetric spaces. The surprising corollary is that if the maximal compact subgroups of G_0 are of rank equal to that of G_0 , then the open G_0 -orbits on X are indefinite-kaehler symmetric spaces.

NOTES FOR CHAPTER II. From August 1965 to May 1968, this author had a number of conversations with Bertram Kostant in which we tried to find the structure of the holomorphic arc components. We succeeded in the integrable case and saw that the normalizer of $\mathcal{O}_x^u \cap \tau\mathcal{O}_x^u$ should play a role. Statement 1 of Theorem 8.5 is due to Kostant, and the general case as worked out in §8 was undoubtedly guided by the background of discussions with him.

Lemma 9.22 and Corollary 9.25 are due to Kostant.

Except for the basic contribution of Kostant described above, the material of §8 and §9 was worked out by this author.

8. Holomorphic arc components. Given a complex flag manifold $X = G/P$ and a real form G_0 of G , the open G_0 -orbits on X are complex submanifolds. If $G_0(x) \subset X$ is the Borel embedding of an Hermitian symmetric space of noncompact type, then [15] one also knows that

every G_0 -orbit on the boundary of $G_0(x)$ has a natural decomposition into complex submanifolds of X . In this section we give an analogous decomposition of an arbitrary G_0 -orbit into "holomorphic arc components," including an explicit criterion and characterization of the cases where those components are complex submanifolds of X .

Let D be a subset of a complex analytic space V . By **holomorphic arc in D** we mean a holomorphic map

$$f: \{z \in \mathbf{C}: |z| < 1\} \rightarrow V$$

of the unit disc into V such that D contains the image of f . By **chain of holomorphic arcs in D** we mean a finite sequence $\{f_1, \dots, f_k\}$ of holomorphic arcs in D such that the image of f_i meets the image of f_{i+1} for $1 \leq i < k$.

8.1. DEFINITION. Let D be a subset of a complex analytic space V . Then the **holomorphic arc components** of D in V are the equivalence classes of elements of D under the relation: $v_1 \sim v_2$ if there is a chain $\{f_1, \dots, f_k\}$ of holomorphic arcs in D such that v_1 is in the image of f_1 and v_2 is in the image of f_k .

We draw some simple consequences from this definition.

8.2. LEMMA. Let V be a complex analytic space, let A be a group of holomorphic diffeomorphisms of V , and let D be an A -stable subset of V . Let S be a holomorphic arc component of D in V and define $N_A(S) = \{a \in A: a(S) = S\}$.

1. If $a \in A$ and $a(S)$ meets S , then $a(S) = S$ i.e. $a \in N_A(S)$.
2. If D is an A -orbit then $N_A(S)$ is transitive on S .
3. If A is a Lie transformation group on V , and if D is an A -orbit, then $N_A(S)$ is a Lie subgroup of A , and S and D are embedded real submanifolds of V .

PROOF. For (1) we have $s_1, s_2 \in S$ with $a(s_1) = s_2$. If $s \in S$ we have a chain $\{f_1, \dots, f_k\}$ of holomorphic arcs in D from s_1 to s ; now $\{a \cdot f_1, \dots, a \cdot f_k\}$ is a chain of holomorphic arcs in D from $a(s_1) = s_2$ to $a(s)$; thus $a(s) \in S$; that proves $a(S) \subset S$. Similarly $a^{-1}(S) \subset S$. Thus $a(S) = S$.

For (2) let $s_1, s_2 \in S$. As $s_i \in D$ we have $a \in A$ with $a(s_1) = s_2$. Now $a \in N_A(S)$ by (1). Thus $N_A(S)$ has an element that sends s_1 to s_2 .

For (3) we first note that S is arcwise connected. Lifting arcs via $N_A(S) \rightarrow S$ we see that the identity component $N_A(S)^0$ is arcwise connected and is transitive on S ; thus $N_A(S)^0$ is an analytic subgroup of A , so $N_A(S)$ is a Lie subgroup, and S is an embedded real submanifold of the embedded submanifold D . Q.E.D.

8.3. NOTATION. Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , and $x \in X$. Denote

(8.3a) $S_{[x]}$: holomorphic arc component of $G_0(x)$ in X that contains x .

(8.3b) $N_{[x],0}$: identity component of $\{g \in G_0: gS_{[x]} = S_{[x]}\}$.

(8.3c) $\mathfrak{N}_{[x],0}$ is the Lie algebra of $N_{[x],0}$, and $\mathfrak{N}_{[x]} = \mathfrak{N}_{[x],0}^{\mathbb{C}}$.

(8.3d) $N_{[x]}$ is the complex analytic subgroup of G for $\mathfrak{N}_{[x]}$.

Lemma 8.2 says that $S_{[x]}$ is an embedded real submanifold of X , $N_{[x],0}$ is a Lie subgroup of G_0 , and $N_{[x],0}$ is transitive on $S_{[x]}$. Note $\tau\mathfrak{N}_{[x]} = \mathfrak{N}_{[x]}$ by construction.

8.4. LEMMA. Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , $x \in X$, $\mathfrak{H} \subset \mathfrak{P}_x$ a τ -stable Cartan subalgebra of \mathfrak{G} , and \mathfrak{L} a complex subalgebra of \mathfrak{G} such that

$$\tau\mathfrak{L} = \mathfrak{L}, \quad [\mathfrak{H}, \mathfrak{L}] \subset \mathfrak{L} \quad \text{and} \quad \mathfrak{L} \subset \mathfrak{P}_x + \tau\mathfrak{P}_x.$$

Let $\mathfrak{L}_0 = \mathfrak{L} \cap \mathfrak{G}_0$, real form of \mathfrak{L} ; let L and L_0 denote the analytic subgroups of G and G_0 for \mathfrak{L} and \mathfrak{L}_0 .

1. $L_0(x)$ and $L(x)$ are complex submanifolds of X with $L_0(x)$ open in $L(x)$.

2. $L_0(x) \subset S_{[x]}$.

3. $\mathfrak{L}_0 \subset \mathfrak{N}_{[x],0}$ and $L_0 \subset N_{[x],0}$; $\mathfrak{L} \subset \mathfrak{N}_{[x]}$ and $L \subset N_{[x]}$.

PROOF. Choice of \mathfrak{H} specifies \mathfrak{P}_x^r and $\mathfrak{P}_x^{\pm u}$. As \mathfrak{L} is complex and normalized by \mathfrak{H} , it is $\mathfrak{L} \cap \mathfrak{H}$ plus certain root spaces; now $\mathfrak{L} \subset (\mathfrak{P}_x + \tau\mathfrak{P}_x)$ shows

$$\mathfrak{L} = \mathfrak{L}^0 + \mathfrak{L}^+ + \mathfrak{L}^-, \quad \mathfrak{L}^0 = \mathfrak{L} \cap (\mathfrak{P}_x \cap \tau\mathfrak{P}_x), \quad \mathfrak{L}^+ = \mathfrak{L} \cap \tau\mathfrak{P}_x^{-u} \subset \mathfrak{P}_x$$

and (using $\tau\mathfrak{L} = \mathfrak{L}$)

$$\mathfrak{L}^- = \mathfrak{L} \cap \mathfrak{P}_x^{-u} \subset \tau\mathfrak{P}_x \quad \text{with} \quad \mathfrak{L}^- = \tau\mathfrak{L}^+.$$

$L(x)$ is a complex submanifold of X because L is a complex Lie subgroup of G . Now

$$\begin{aligned} \dim_{\mathbb{R}} L_0(x) &= \dim_{\mathbb{R}} \mathfrak{L}_0 - \dim_{\mathbb{R}} \mathfrak{L}_0 \cap \mathfrak{P}_x \\ &= \dim_{\mathbb{R}} \mathfrak{L}_0 - \dim_{\mathbb{R}} \mathfrak{L}_0 \cap \mathfrak{P}_x \cap \tau\mathfrak{P}_x \\ &= \dim_{\mathbb{C}} \mathfrak{L} - \dim_{\mathbb{C}} \mathfrak{L}^0 \\ &= \dim_{\mathbb{C}} \mathfrak{L}^+ + \dim_{\mathbb{C}} \mathfrak{L}^- = 2 \dim_{\mathbb{C}} \mathfrak{L}^- \\ &= 2 \{ \dim_{\mathbb{C}} \mathfrak{L} - \dim_{\mathbb{C}} \mathfrak{L} \cap \mathfrak{P}_x \} \\ &= 2 \dim_{\mathbb{C}} L(x) = \dim_{\mathbb{R}} L(x) \end{aligned}$$

shows that $L_0(x)$ is open in $L(x)$. That proves (1).

Covering by complex local coordinate neighborhoods one sees that $L_0(x)$ has just one holomorphic arc component. Thus $L_0(x) \subset S_{[x]}$, proving (2), and (3) follows immediately. Q.E.D.

We use Lemma 8.4 to prove $\mathfrak{N}_{[x]}$ parabolic by constructing a parabolic subalgebra $\mathcal{Q}_{[x]} \subset \mathfrak{N}_{[x]}$ of \mathfrak{g} . That is the first step in the determination of $S_{[x]}$ in algebraic terms.

8.5. THEOREM. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , $x \in X$, $\mathfrak{K} \subset \mathcal{P}_x$ a τ -stable Cartan subalgebra of \mathfrak{g} , and Φ a subset of a simple \mathfrak{K} -root system such that $\mathcal{P}_x = \mathcal{P}_\Phi$.*

Define a τ -stable linear form δ_Φ on \mathfrak{K} by

$$(8.6a) \quad \delta_\Phi = \sum_{\varphi \in \Phi} \varphi.$$

Define τ -stable sets of \mathfrak{K} -roots of \mathfrak{g} by

$$(8.6b) \quad \Gamma^r = \{ \varphi \in \Delta : \langle \varphi, \delta_\Phi \rangle = 0 \}, \quad \Gamma^u = \{ \varphi \in \Delta : \langle \varphi, \delta_\Phi \rangle > 0 \},$$

$$\Gamma^* = \Gamma^r \cup \Gamma^u.$$

Define a τ -stable parabolic subalgebra $\mathcal{Q}_{[x]} \subset \mathfrak{g}$ by

$$(8.6c) \quad \mathcal{Q}_{[x]}^r = \mathfrak{K} + \sum_{\Gamma^r} \mathfrak{G}_\varphi, \quad \mathcal{Q}_{[x]}^u = \sum_{\Gamma^u} \mathfrak{G}_\varphi, \quad \mathcal{Q}_{[x]} = \mathcal{Q}_{[x]}^r + \mathcal{Q}_{[x]}^u.$$

1. *The normalizer $\mathfrak{N}_{\mathfrak{g}}(\mathfrak{u}_x)$ of $\mathfrak{u}_x = \mathcal{P}_x^u \cap \tau \mathcal{P}_x^u$ satisfies $\mathfrak{N}_{\mathfrak{g}}(\mathfrak{u}_x)^r \subset \mathcal{Q}_{[x]}^r$ and $\mathfrak{N}_{\mathfrak{g}}(\mathfrak{u}_x)^u \subset \mathcal{Q}_{[x]}^u$.*
2. *$\mathfrak{u}_x \subset \mathfrak{N}_{\mathfrak{g}}(\mathfrak{u}_x) \subset \mathcal{Q}_{[x]} \subset \{ \mathfrak{N}_{[x]} \cap (\mathcal{P}_x + \tau \mathcal{P}_x) \}$.*
3. *$\mathfrak{N}_{[x]}$ is a parabolic subalgebra of \mathfrak{g} and $N_{[x],0}$ is the identity component of the parabolic subgroup $N_{[x]} \cap G_0$ of G_0 .*

PROOF. Note $\mathfrak{K} + \mathfrak{u}_x \subset \mathfrak{N}_{\mathfrak{g}}(\mathfrak{u}_x)$, so $\mathfrak{N}_{\mathfrak{g}}(\mathfrak{u}_x)$ is the sum of \mathfrak{K} with certain root spaces. Let φ be a root with $\mathfrak{G}_\varphi \subset \mathfrak{N}_{\mathfrak{g}}(\mathfrak{u}_x)$, let $\mathfrak{g}[\varphi]$ denote the three dimensional simple algebra $h_\varphi \mathbb{C} + \mathfrak{G}_\varphi + \mathfrak{G}_{-\varphi}$, and let \mathfrak{Z} denote the centralizer of $\mathfrak{g}[\varphi]$ in \mathfrak{g} . Then

$$\mathfrak{Z} = \{ h \in \mathfrak{K} : \varphi(h) = 0 \} + \sum_{\psi \perp \perp \varphi} \mathfrak{G}_\psi$$

where $\psi \perp \perp \varphi$ denotes strong orthogonality, i.e. denotes that neither of $\varphi \pm \psi$ is a root. And further

$$\mathfrak{g} = \mathfrak{Z} + \sum \mathfrak{N}_i, \quad \mathfrak{N}_1 = \mathfrak{g}[\varphi],$$

where the \mathfrak{N}_i are the nontrivial irreducible $\mathfrak{g}[\varphi]$ -modules under $\text{ad}_{\mathfrak{g}}$. Thus for $i \neq 1$ we have \mathfrak{N}_i of the form

$$\mathfrak{M}_i = \sum_{j=-r}^s \mathfrak{G}_{\psi+j\varphi}, \quad \psi \in \Delta, \quad \frac{2\langle \psi, \varphi \rangle}{\langle \varphi, \varphi \rangle} = r - s,$$

for the maximal string $S_\psi = \{\psi + j\varphi\}_{-r \leq j \leq s}$ of roots.

Suppose that S_ψ meets $\Phi^u \cap \tau\Phi^u$, say $\psi + j_0\varphi \in \Phi^u \cap \tau\Phi^u$. Then $\mathfrak{G}_\varphi \subset \mathfrak{N}_\mathfrak{g}(\mathfrak{u}_x)$ implies $\psi + j\varphi \in \Phi^u \cap \tau\Phi^u$ for $j_0 \leq j \leq s$, so we have an integer j_ψ , $-r \leq j_\psi \leq s$, such that $S_\psi \cap (\Phi^u \cap \tau\Phi^u) = \{\psi + j\varphi : j_\psi \leq j \leq s\}$. Note $S_\psi \subset \Phi^u \cap \tau\Phi^u$ if and only if $j_\psi = -r$. But $j' > j''$ implies $\langle \varphi, \psi + j'\varphi \rangle > \langle \varphi, \psi + j''\varphi \rangle$ because $\langle \varphi, \varphi \rangle > 0$; and $\text{ad}(h_\varphi)$ has trace $0 = \langle \varphi, \sum_{j=-r}^s (\psi + j\varphi) \rangle$ on \mathfrak{M}_i because $\mathfrak{g}[\varphi]$ is semisimple; thus

either $S_\psi \subset \Phi^u \cap \tau\Phi^u$ and $\langle \varphi, \sum_{S_\psi \cap \Phi^u \cap \tau\Phi^u} \gamma \rangle = 0$

or $S_\psi \not\subset \Phi^u \cap \tau\Phi^u$ and $\langle \varphi, \sum_{S_\psi \cap \Phi^u \cap \tau\Phi^u} \gamma \rangle > 0$.

On the other hand, $\mathfrak{G}_{-\varphi} \subset \mathfrak{N}_\mathfrak{g}(\mathfrak{u}_x)$ if and only if $\mathfrak{G}_{-\varphi}$ normalizes each $\mathfrak{u}_x \cap \mathfrak{M}_i$, and, as above, that is equivalent to $S_\psi \subset \Phi^u \cap \tau\Phi^u$ whenever S_ψ meets $\Phi^u \cap \tau\Phi^u$. Thus for $\varphi \notin \Phi^u \cap \tau\Phi^u$, we have $\langle \varphi, \delta_\Phi \rangle = 0$ if $\mathfrak{G}_\varphi \subset \mathfrak{N}_\mathfrak{g}(\mathfrak{u}_x)^r$ and $\langle \varphi, \delta_\Phi \rangle > 0$ if $\mathfrak{G}_\varphi \subset \mathfrak{N}_\mathfrak{g}(\mathfrak{u}_x)^u$. And for φ in $\Phi^u \cap \tau\Phi^u$ we have $\mathfrak{G}_\varphi \subset \mathfrak{N}_\mathfrak{g}(\mathfrak{u}_x)^u$ with $\langle \varphi, \delta_\Phi \rangle > 0$. That proves $\mathfrak{N}_\mathfrak{g}(\mathfrak{u}_x)^u \subset \mathcal{Q}_{[x]}^u$ and $\mathfrak{N}_\mathfrak{g}(\mathfrak{u}_x)^r \subset \mathcal{Q}_{[x]}^r$.

We have just proved $\mathfrak{u}_x \subset \mathfrak{N}_\mathfrak{g}(\mathfrak{u}_x) \subset \mathcal{Q}_{[x]}$. And $\mathfrak{u}_x \subset \mathfrak{N}_\mathfrak{g}(\mathfrak{u}_x)^u \subset \mathcal{Q}_{[x]}^u$ proves $\mathcal{Q}_{[x]} \subset \mathcal{P}_x + \tau\mathcal{P}_x$. As $\mathfrak{K} \subset \mathcal{Q}_{[x]} = \tau\mathcal{Q}_{[x]}$, Lemma 8.4 says $\mathcal{Q}_{[x]} \subset \mathfrak{N}_{[x]}$. That proves (2), and (3) follows because $\mathcal{Q}_{[x]}$ is parabolic in \mathfrak{g} . Q.E.D.

The next step in the algebraic determination of S consists of expanding $\mathcal{Q}_{[x]}$ out toward $\mathfrak{N}_{[x]}$ as follows.

8.7. PROPOSITION. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , $x \in X$, $\mathfrak{K} \subset \mathcal{P}_x$ a τ -stable Cartan subalgebra of \mathfrak{g} , and Φ a subset of a simple \mathfrak{K} -root system such that $\mathcal{P}_x = \mathcal{P}_\Phi$. Retain the notation (8.6). Define a τ -stable subset $\Gamma^0 \subset (\Phi^* \cup \tau\Phi^*) \setminus \Gamma^*$ by*

$$(8.8a) \quad \Gamma^0 = \{ \varphi \in \Delta : -\varphi \notin \Phi^u \cap \tau\Phi^u, \langle \varphi, \delta_\Phi \rangle < 0, \varphi + \tau\varphi \notin \Delta \}.$$

Define a τ -stable subspace $\mathfrak{M}_{[x]}$ of \mathfrak{g}_0 by

$$(8.8b) \quad \mathfrak{M}_{[x]} = \mathcal{Q}_{[x]} + \sum_{\Gamma^0} \mathfrak{G}_\varphi.$$

1. $\mathcal{Q}_{[x]} \subset \mathfrak{M}_{[x]} \subset \mathfrak{N}_{[x]} \cap (\mathcal{P}_x + \tau\mathcal{P}_x)$.
2. If $\varphi \in \Gamma^0$, then $-\varphi \neq \tau\varphi \neq \varphi$, and either
 - (2a) $\varphi \perp \tau\varphi$, $\tau\varphi - \varphi \in \Delta$, and $h_\varphi\mathfrak{C} + h_{\tau\varphi}\mathfrak{C} + \mathfrak{G}_\varphi + \mathfrak{G}_{-\varphi} + \mathfrak{G}_{\tau\varphi} + \mathfrak{G}_{-\tau\varphi}$ is a τ -stable subalgebra of type $A_1 \oplus A_1$ of \mathfrak{g} contained in $\mathfrak{N}_{[x]} \cap (\mathcal{P}_x + \tau\mathcal{P}_x)$; or
 - (2b) $\langle \varphi, \tau\varphi \rangle = \frac{1}{2} \langle \varphi, \varphi \rangle$, $\tau\varphi - \varphi \in \Delta$, and $h_\varphi\mathfrak{C} + h_{\tau\varphi}\mathfrak{C} + \mathfrak{G}_\varphi + \mathfrak{G}_{-\varphi} + \mathfrak{G}_{\tau\varphi} + \mathfrak{G}_{-\tau\varphi} + \mathfrak{G}_{\tau\varphi - \varphi} + \mathfrak{G}_{\varphi - \tau\varphi}$ is a τ -stable subalgebra of type A_2 in \mathfrak{g} contained in $\mathfrak{N}_{[x]} \cap (\mathcal{P}_x + \tau\mathcal{P}_x)$.
3. $\mathfrak{N}_{[x]} \subset \mathcal{P}_x + \tau\mathcal{P}_x$ if and only if $\mathfrak{N}_{[x]} = \mathfrak{M}_{[x]}$.

PROOF. Decompose the real form $\mathfrak{K}_0 = \mathfrak{K} \cap \mathfrak{G}_0$ of \mathfrak{K} as $\mathfrak{K}_0 = \mathfrak{K}_T + \mathfrak{K}_V$ such that $i\mathfrak{K}_T + \mathfrak{K}_V$ is the real span of the roots. If $\varphi \in \Delta$ then $\varphi = \varphi_T + \varphi_V$ where $\varphi_T(\mathfrak{K}_V) = 0 = \varphi_V(\mathfrak{K}_T)$, and then $\tau\varphi = -\varphi_T + \varphi_V$. Now suppose $\langle \varphi, \delta_\Phi \rangle < 0$ with $-\varphi \notin \Phi^u \cap \tau\Phi^u$. If $\tau\varphi = \varphi$ then $\mathfrak{G}_\varphi \subset \mathcal{P}_x + \tau\mathcal{P}_x$ implies $\mathfrak{G}_\varphi \subset \mathcal{P}_x \cap \tau\mathcal{P}_x \subset \mathfrak{N}_\mathfrak{G}(\mathfrak{U}_x)$, so $\langle \varphi, \delta_\Phi \rangle \geq 0$, contradicting $\langle \varphi, \delta_\Phi \rangle < 0$. If $\tau\varphi = -\varphi$ then $\langle \tau\varphi, \delta_\Phi \rangle = \langle \tau\varphi, \tau\delta_\Phi \rangle = \langle \varphi, \delta_\Phi \rangle = -\langle \tau\varphi, \delta_\Phi \rangle$, contradicting $\langle \varphi, \delta_\Phi \rangle < 0$. Thus $-\varphi \neq \tau\varphi \neq \varphi$, i.e. $\varphi_T \neq 0 \neq \varphi_V$. In particular $|\langle \varphi, \tau\varphi \rangle| < |\langle \varphi, \varphi \rangle|$ so the integer $2\langle \varphi, \tau\varphi \rangle / \langle \varphi, \varphi \rangle = 2\langle \varphi, \tau\varphi \rangle / \langle \tau\varphi, \tau\varphi \rangle$ is -1 , 0 or $+1$.

Suppose further that $\varphi + \tau\varphi$ is not a root, i.e. that $\varphi \in \Gamma^0$. Then the maximal chain $\{\varphi + j\tau\varphi\}_{-r \leq j \leq s}$ of roots has $s = 0$, so $2\langle \varphi, \tau\varphi \rangle / \langle \tau\varphi, \tau\varphi \rangle = r \geq 0$, proving that $2\langle \varphi, \tau\varphi \rangle / \langle \varphi, \varphi \rangle$ is 0 (in which case $\varphi \perp \tau\varphi$ and $\tau\varphi - \varphi \in \Delta$) or is 1 (in which case $\langle \varphi, \tau\varphi \rangle = \frac{1}{2}\langle \varphi, \varphi \rangle$, $\tau\varphi - \varphi \in \Delta$, and $\tau\varphi - 2\varphi \in \Delta$). Let \mathfrak{L} denote $h_\varphi\mathbf{C} + h_{\tau\varphi}\mathbf{C} + \mathfrak{G}_\varphi + \mathfrak{G}_{-\varphi} + \mathfrak{G}_{\tau\varphi} + \mathfrak{G}_{-\tau\varphi}$ in the first case, $h_\varphi\mathbf{C} + h_{\tau\varphi}\mathbf{C} + \mathfrak{G}_\varphi + \mathfrak{G}_{-\varphi} + \mathfrak{G}_{\tau\varphi} + \mathfrak{G}_{-\tau\varphi} + \mathfrak{G}_{\varphi - \tau\varphi} + \mathfrak{G}_{\tau\varphi - \varphi}$ in the second case. Then \mathfrak{L} is a τ -stable subalgebra of type $A_1 \oplus A_1$ or A_2 in \mathfrak{G} , $[\mathfrak{K}, \mathfrak{L}] \subset \mathfrak{L}$, and $\mathfrak{L} \subset \mathcal{P}_x + \tau\mathcal{P}_x$; the latter is obvious in the first case and follows from $\langle \tau\varphi - \varphi, \delta_\Phi \rangle = 0 = \langle \varphi - \tau\varphi, \delta_\Phi \rangle$ in the second case. Lemma 8.4 says $\mathfrak{L} \subset \mathfrak{N}_{[x]}$. That completes the proof of (2), and (1) is an immediate consequence.

If $\mathfrak{N}_{[x]} = \mathfrak{M}_{[x]}$, then $\mathfrak{M}_{[x]} \subset \mathcal{P}_x + \tau\mathcal{P}_x$ implies $\mathfrak{N}_{[x]} \subset \mathcal{P}_x + \tau\mathcal{P}_x$. Conversely suppose $\mathfrak{N}_{[x]} \subset \mathcal{P}_x + \tau\mathcal{P}_x$. Then $\mathfrak{M}_{[x]} \subset \mathfrak{N}_{[x]}$ implies that $\mathfrak{N}_{[x]} = \mathfrak{M}_{[x]} + \sum_A \mathfrak{G}_\varphi$ where A is some τ -stable set of roots such that

$$\text{if } \alpha \in A \text{ then } -\alpha \notin \Phi^u \cap \tau\Phi^u, \quad \langle \alpha, \delta_\Phi \rangle < 0 \text{ and } \alpha + \tau\alpha \in \Delta.$$

If A is not empty, choose $\alpha \in A$ and observe that $\mathfrak{G}_{\alpha + \tau\alpha} \subset [\mathfrak{N}_{[x]}, \mathfrak{N}_{[x]}] \subset \mathfrak{N}_{[x]} \subset \mathcal{P}_x + \tau\mathcal{P}_x$, so $\tau(\alpha + \tau\alpha) = \alpha + \tau\alpha$ implies $\mathfrak{G}_{\alpha + \tau\alpha} \subset \mathcal{P}_x \cap \tau\mathcal{P}_x$; that implies $\langle \alpha + \tau\alpha, \delta_\Phi \rangle \geq 0$, contradicting

$$\langle \alpha + \tau\alpha, \delta_\Phi \rangle = \langle \alpha, \delta_\Phi \rangle + \langle \tau\alpha, \tau\delta_\Phi \rangle = 2\langle \alpha, \delta_\Phi \rangle < 0.$$

Thus A is empty and $\mathfrak{N}_{[x]} = \mathfrak{M}_{[x]}$. Q.E.D.

We apply Theorem 8.5 and Proposition 8.7 to the question of whether holomorphic arc components of $G_0(x)$ are complex submanifolds of X . Note that, in Theorem 8.9, (5) is the algebraic criterion for (1); it then specifies $\mathfrak{N}_{[x]}$ by (4), thus specifying $S_{[x]} = N_{[x],0}(x) \cong N_{[x],0}/(P_x \cap \tau P_x)_0$.

8.9. THEOREM. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , $x \in X$, $\mathfrak{K} \subset \mathcal{P}_x$ a τ -stable Cartan subalgebra of \mathfrak{G} , and Φ a subset of a simple \mathfrak{K} -root system such that $\mathcal{P}_x = \mathcal{P}_\Phi$. Retain the notation (8.6) and (8.8). Then the following conditions are equivalent:*

1. Every holomorphic arc component of $G_0(x)$ in X is a complex submanifold of X .

2. $S_{[z]}$ is a complex submanifold of X .
3. $\mathfrak{N}_{[z]} \subset \mathcal{O}_z + \tau\mathcal{O}_z$.
4. $\mathfrak{N}_{[z]} = \mathfrak{M}_{[z]}$, given by (8.6) and (8.8).
5. $\mathfrak{M}_{[z]}$ is an algebra.

PROOF. (1) implies its special case (2). Conversely, if $g \in G_0$, then $gS_{[z]} = S_{[gz]}$, so (2) implies (1).

Lemma 8.4 says that (3) implies (2). For the converse we define a G_0 -invariant distribution $T = \{T_y\}_{y \in G_0(x)}$ on $G_0(x)$ as follows. If $v \in \mathfrak{G}$ then v^* denotes the corresponding real tangent vector field on X , and its value at $y \in X$ is denoted v_y^* . Now define, for every $y \in G_0(x)$,

$$(8.10a) \quad T_y = \{v_y^* : v \in (\mathcal{O}_y + \tau\mathcal{O}_y) \cap \mathfrak{G}_0\}, \text{ real subspace of } G_0(x)_y.$$

We will prove that every holomorphic arc in $G_0(x)$ is tangent to T , i.e. that

$$(8.10b) \quad \text{if } f: D \rightarrow X \text{ is a holomorphic arc in } G_0(x) \text{ then } f_*D_z \subset T_{f(z)} \text{ for every } z \in D.$$

Assume (8.10b). As (2) says that each tangent space to $S_{[z]}$ is filled out by the tangent spaces of holomorphic arcs in $G_0(x)$, it follows that $\{v_y^* : v \in \mathfrak{N}_{[y],0}\} \subset T_y$ for every $y \in G_0(x)$; it follows that $\mathfrak{N}_{[z],0} \subset (\mathcal{O}_z + \tau\mathcal{O}_z) \cap \mathfrak{G}_0$, so $\mathfrak{N}_{[z]} \subset \mathcal{O}_z + \tau\mathcal{O}_z$. Thus (2) implies (3) modulo the truth of (8.10b).

We prove (8.10b). Let $z \in D$, $\beta \in D_z$. Let J_D and J_X be the almost complex structures on D and X , so $f_*(J_D\beta) = J_X f_*\beta$ because f is holomorphic. $f(D) \subset G_0(x)$ implies that both $f_*\beta$ and $f_*J_D\beta$ are in $G_0(x)_{f(z)}$. Thus we have $v \in \mathfrak{G}_0$ with $f_*\beta = v_{f(z)}^*$, and $f_*J_D\beta = J_X f_*\beta = J_X v_{f(z)}^* = (iv)_{f(z)}^*$. That says $iv \in \mathfrak{G}_0 + \mathcal{O}_{f(z)}$, say $iv = u + w$ where $u \in \mathfrak{G}_0$ and $w \in \mathcal{O}_{f(z)}$. Now $\tau v = v$ implies

$$2iv = iv - \tau(iv) = (u + w) - (\tau u + \tau w) = w - \tau w.$$

Thus $2iv \in \mathcal{O}_{f(z)} + \tau\mathcal{O}_{f(z)}$, so $v \in \mathfrak{G}_0 \cap (\mathcal{O}_{f(z)} + \tau\mathcal{O}_{f(z)})$, proving $f_*\beta = v_{f(z)}^* \in T_{f(z)}$. (8.10b) is proved.

Proposition 8.7 contains the equivalence of (3) and (4). Summarizing now, we have proved (8.10b) and have shown that (1), (2), (3) and (4) are equivalent. (4) implies (5) because $\mathfrak{N}_{[z]}$ is an algebra. Now we only need to prove that (5) implies (4).

Suppose that $\mathfrak{M}_{[z]}$ is an algebra. Let $\mathfrak{N}_{[z],0}$ denote its real form $\mathfrak{N}_{[z]} \cap \mathfrak{G}_0$; let $M_{[z]}$ and $M_{[z],0}$ denote the analytic subgroups of G and G_0 for $\mathfrak{N}_{[z]}$ and $\mathfrak{N}_{[z],0}$. Define $\mathfrak{N}_{[gz]} = \text{ad}(g)\mathfrak{N}_{[z]}$, then $\mathfrak{N}_{[gz],0}$, $M_{[gz]}$ and $M_{[gz],0}$ as above, for every $g \in G_0$; those algebras and groups are well defined because $M_{[z],0} \supset (P_x \cap \tau P_x)_0 = G_0 \cap P_x$. Now we have a distribution $M = \{M_y\}$ on $G_0(x)$ given by

$$M_y = \{v_y^* : v \in \mathfrak{N}_{[y],0}\}.$$

The distribution is integrable because M_y is the tangent space to $M_{[y],0}(y)$ at y . We will prove that every holomorphic arc in $G_0(x)$ is tangent to M . As M is integrable, and as its integral manifolds $M_{[y],0}(y)$ are complex submanifolds of X by Lemma 8.4 and $M_{[gx],0}(gx) = gM_{[x],0}(x)$ for $g \in G_0$, we will conclude that the holomorphic arc components of $G_0(x)$ are complex submanifolds of X . Then (5) implies (4) and the theorem is proved.

Now we show that every holomorphic arc in $G_0(x)$ is tangent to M . As $M_{g(y)} = g_*M_y$ for $y \in G_0(x)$ and $g \in G_0$, it suffices to consider holomorphic arcs $f: D \rightarrow X$ in $G_0(x)$ with $f(0) = x$, and to prove $f_*D_0 \subset M_x$. If $f_*D_0 = 0$ this is automatic. Now suppose $f_*D_0 \neq 0$. The implicit function theorem says that f maps a neighborhood of 0 diffeomorphically onto its image, so we cut D down and assume that $f: D \rightarrow f(D)$ is a diffeomorphism. Let η be a real vector field on D , let J_D and J_X be the almost complex structures on D and X , and let $\xi = f_*\eta$. Then (8.10b) ξ and $J_X\xi = f_*J_D\eta$ are tangent to $T|_{f(D)}$, and so is $[\xi, J_X\xi] = f_*[\eta, J_D\eta]$.

Given $v \in \mathfrak{G}_0$, the vector field v^* induced on $G_0(x)$ is induced from the 1-parameter group $\{\exp tv\} \subset G_0$ acting on $G_0(x)$; it is the projection of a right invariant vector field on G_0 . Let σ be a smooth local section of $G_0 \rightarrow G_0/(P_x \cap \tau P_x)_0 = G_0(x)$ in a neighborhood W of x . If $v \in \mathfrak{G}_0$, and if \bar{v} denotes the corresponding left invariant vector field on G_0 , then $\bar{v}|_{\sigma(W)}$ projects to a vector field v^\dagger on W . Observe

$$\begin{aligned} T_v &= \{v_y^*: v \in \mathfrak{G}_0 \cap (\mathcal{P}_y + \tau\mathcal{P}_y)\} \text{ is } \{v_y^\dagger: v \in \mathfrak{G}_0 \cap (\mathcal{P}_x + \tau\mathcal{P}_x)\}, \\ M_v &= \{v_y^*: v \in \mathfrak{M}_{[y],0}\} \text{ is } \{v_y^\dagger: v \in \mathfrak{M}_{[x],0}\}, \\ N_v &= \{v_y^*: v \in \mathfrak{N}_{[y],0}\} \text{ is } \{v_y^\dagger: v \in \mathfrak{N}_{[x],0}\}, \end{aligned}$$

from G_0 -invariance of the corresponding distributions. Now cut W down if necessary, so that the tangent vector field ξ to $f(D) \subset S_{[x]} = N_{[x],0}(x)$ has representation

$$\xi_y = \zeta(y)_y^\dagger \text{ where } \zeta: f(D) \cap W \rightarrow \mathfrak{N}_{[x],0} \text{ is smooth.}$$

As ξ is tangent to T , ζ takes values in $(\mathcal{P}_x + \tau\mathcal{P}_x) \cap \mathfrak{N}_{[x],0}$.

Let A' denote the set of all \mathfrak{H} -roots φ such that $\mathfrak{G}_\varphi \subset (\mathcal{P}_x + \tau\mathcal{P}_x) \cap \mathfrak{N}_{[x]}$ but $\mathfrak{G}_\varphi \not\subset \mathfrak{M}_{[x]}$. Then $(\mathcal{P}_x + \tau\mathcal{P}_x) \cap \mathfrak{N}_{[x]} = \mathfrak{M}_{[x]} + \mathfrak{A}$ direct sum where $\mathfrak{A} = \sum_{A'} \mathfrak{G}_\varphi$, and ζ takes values in $\mathfrak{M}_{[x],0} + \mathfrak{A}_0$ where $\mathfrak{A}_0 = \mathfrak{A} \cap \mathfrak{G}_0$. Decompose $\zeta = \mu + \alpha$ where μ takes values in $\mathfrak{M}_{[x],0}$ and α takes values in \mathfrak{A}_0 . That decomposes $\xi = m + a$ and $J_X\xi = J_Xm + J_Xa$ where

$$m_y = \mu(y)_y^\dagger \text{ and } a_y = \alpha(y)_y^\dagger.$$

Now

$$\begin{aligned}
 [a, J_X a]_x &= [\xi - m, J_X \xi - J_X m]_x \\
 &= [\xi, J_X \xi]_x - [m, J_X \xi]_x - [\xi, J_X m]_x + [m, J_X m]_x.
 \end{aligned}$$

We have already seen $[\xi, J_X \xi]_x \in T_x$. We have $[m, J_X \xi]_x \in T_x$ and $[\xi, J_X m]_x \in T_x$ because m and $J_X m$ are tangent to M and $M_{[x],0}$ preserves the real span of the tangent spaces to holomorphic arcs. And finally $[m, J_X m]_x \in M_x \subset T_x$ because $\mathfrak{M}_{[x],0}$ is an algebra. Thus $[a, J_X a]_x \in T_x$.

$A' = A \cup \tau A$ disjoint where $A = A' \cap \tau \Phi^* = A' \cap -\Phi^u$. If $\varphi \in A$ choose $0 \neq e_\varphi \in \mathfrak{G}_\varphi$. Define $p_\varphi = e_\varphi + \tau e_\varphi \in \mathfrak{G}_0$ and $q_\varphi = i e_\varphi + \tau(i e_\varphi) = i e_\varphi - i \tau e_\varphi \in \mathfrak{G}_0$. As \mathfrak{G}_x^u is the holomorphic tangent space to X at x , $J_X p_\varphi^\dagger = q_\varphi^\dagger$ and $J_X q_\varphi^\dagger = -p_\varphi^\dagger$, for $\varphi \in A$, at x . Invariance then gives $J_X p_\varphi^\dagger = q_\varphi^\dagger$ and $J_X q_\varphi^\dagger = -p_\varphi^\dagger$ on $G_0(x)$ for $\varphi \in A$. Now express $\alpha(y) = \sum_{\varphi \in A} (s_\varphi(y) p_\varphi + t_\varphi(y) q_\varphi)$, so that $a = \sum_{\varphi \in A} (s_\varphi p_\varphi^\dagger + t_\varphi q_\varphi^\dagger)$ and $J_X a = \sum_{\varphi \in A} (-t_\varphi p_\varphi^\dagger + s_\varphi q_\varphi^\dagger)$. Then we compute $[a, J_X a] = \sum_{\varphi, \psi \in A} [s_\varphi^\dagger p_\varphi + t_\varphi q_\varphi^\dagger, -t_\psi p_\psi^\dagger + s_\psi q_\psi^\dagger]$ and

$$\begin{aligned}
 [s_\varphi p_\varphi^\dagger + t_\varphi q_\varphi^\dagger, -t_\psi p_\psi^\dagger + s_\psi q_\psi^\dagger] &= -s_\varphi \cdot p_\varphi^\dagger(t_\psi) \cdot p_\psi^\dagger + t_\psi \cdot p_\psi^\dagger(s_\varphi) \cdot p_\varphi^\dagger - s_\varphi t_\psi [p_\varphi, p_\psi]^\dagger \\
 &\quad + s_\varphi \cdot p_\varphi^\dagger(s_\psi) \cdot q_\psi^\dagger - s_\psi \cdot q_\psi^\dagger(s_\varphi) \cdot p_\varphi^\dagger + s_\varphi s_\psi [p_\varphi, q_\psi]^\dagger \\
 &\quad - t_\varphi \cdot q_\varphi^\dagger(t_\psi) \cdot p_\psi^\dagger + t_\psi \cdot p_\psi^\dagger(t_\varphi) \cdot q_\varphi^\dagger - t_\varphi t_\psi [q_\varphi, p_\psi]^\dagger \\
 &\quad + t_\varphi \cdot q_\varphi^\dagger(s_\psi) \cdot q_\psi^\dagger - s_\psi \cdot q_\psi^\dagger(t_\varphi) \cdot q_\varphi^\dagger + t_\varphi s_\psi [q_\varphi, q_\psi]^\dagger.
 \end{aligned}$$

As $\lambda \in A$ implies $p_\lambda^\dagger, q_\lambda^\dagger \in T_x$, and as $[a, J_X a]_x \in T_x$, it follows that

$$\begin{aligned}
 0 &\equiv [a, J_X a]_x \text{ mod. } T_x \\
 &\equiv \sum_{\varphi, \psi \in A} \{ -s_\varphi(x) t_\psi(x) [p_\varphi, p_\psi]_x^\dagger + s_\varphi(x) s_\psi(x) [p_\varphi, q_\psi]_x^\dagger \\
 &\quad - t_\varphi(x) t_\psi(x) [q_\varphi, p_\psi]_x^\dagger + t_\varphi(x) s_\psi(x) [q_\varphi, q_\psi]_x^\dagger \} \text{ mod. } T_x \\
 &= \sum_{\varphi, \psi \in A} \{ -s_\varphi(x) t_\psi(x) ([e_\varphi, e_\psi] + \tau[e_\varphi, e_\psi] + [e_\varphi, \tau e_\psi] + [\tau e_\varphi, e_\psi])_x^\dagger \\
 &\quad + s_\varphi(x) s_\psi(x) (i[e_\varphi, e_\psi] - i\tau[e_\varphi, e_\psi] - i[e_\varphi, \tau e_\psi] + i[\tau e_\varphi, e_\psi])_x^\dagger \\
 &\quad - t_\varphi(x) t_\psi(x) (i[e_\varphi, e_\psi] - i\tau[e_\varphi, e_\psi] + i[e_\varphi, \tau e_\psi] - i[\tau e_\varphi, e_\psi])_x^\dagger \\
 &\quad + t_\varphi(x) s_\psi(x) (-[e_\varphi, e_\psi] - \tau[e_\varphi, e_\psi] + [e_\varphi, \tau e_\psi] + [\tau e_\varphi, e_\psi])_x^\dagger \} \\
 &= -2 \sum_{\varphi, \psi \in A} s_\varphi(x) t_\psi(x) ([e_\varphi, \tau e_\psi] + [\tau e_\varphi, e_\psi])_x^\dagger \\
 &\quad + \sum_{\varphi, \psi \in A} \{ s_\varphi(x) s_\psi(x) + t_\varphi(x) t_\psi(x) \} \{ i([\tau e_\varphi, e_\psi] + [\tau e_\psi, e_\varphi])_x^\dagger \}.
 \end{aligned}$$

Let $\beta \in A$. Then $\langle \beta, \delta_\Phi \rangle < 0$, $-\beta \notin \Phi^u \cap \tau\Phi^u$ and (because $\mathfrak{g}_\beta \subset \mathfrak{M}_{[x]}$) $\beta + \tau\beta$ is a root. If $\mathfrak{g}_{\beta+\tau\beta} \subset \mathcal{O}_x + \tau\mathcal{O}_x$, then $\tau\mathfrak{g}_{\beta+\tau\beta} = \mathfrak{g}_{\beta+\tau\beta}$ would imply $\mathfrak{g}_{\beta+\tau\beta} \subset \mathcal{O}_x \cap \tau\mathcal{O}_x$; that would say $\langle \beta + \tau\beta, \delta_\Phi \rangle \geq 0$, contradicting $\langle \beta + \tau\beta, \delta_\Phi \rangle = 2\langle \beta, \delta_\Phi \rangle < 0$. Thus $\mathfrak{g}_{\beta+\tau\beta} \subset \mathcal{O}_x^{-u} \cap \tau\mathcal{O}_x^{-u}$. Define $E_{\beta+\tau\beta} = \{v_x^\dagger: v_x \in \mathfrak{g}_0 \cap \mathfrak{g}_{\beta+\tau\beta}\}$, 2-dimensional real subspace of the tangent space to $G_0(x)$ at x . Then the projection of $[a, J_x a]_x$ to $E_{\beta+\tau\beta}$ is

$$\begin{aligned} 0 &= \{s_\beta(x)^2 + t_\beta(x)^2\} \{2i[\tau e_\beta, e_\beta]\}_x^\dagger \\ &+ \sum_{\substack{\varphi \neq \psi \text{ in } A \\ \varphi + \tau\psi = \beta + \tau\beta}} (\{is_\varphi(x)s_\psi(x) + it_\varphi(x)t_\psi(x) - 2s_\varphi(x)t_\psi(x)\} [\tau e_\varphi, e_\psi])_x^\dagger \\ &+ \sum_{\substack{\varphi \neq \psi \text{ in } A \\ \varphi + \tau\psi = \beta + \tau\beta}} (\{is_\varphi(x)s_\psi(x) + it_\varphi(x)t_\psi(x) + 2s_\varphi(x)t_\psi(x)\} [\tau e_\psi, e_\varphi])_x^\dagger. \end{aligned}$$

Suppose that the holomorphic arc f does not satisfy $f_*D_0 \subset M_x$. Then we can choose η such that $\xi_x \notin M_x$, i.e. such that $\alpha(x) \neq 0$, so there is a root $\beta \in A$ for which $s_\beta(x)^2 + t_\beta(x)^2 \neq 0$. Recall that \mathfrak{Q} is disjoint from the parabolic subalgebra $\mathfrak{Q}_{[x]}$, in fact that $\mathfrak{Q} \subset \mathfrak{Q}_{[x]}^u$. So there is an ordering of the \mathfrak{R} -roots of \mathfrak{g} such that A consists of negative roots. Now choose β to be the highest root in A such that $s_\beta(x)^2 + t_\beta(x)^2 \neq 0$, i.e. such that the real numbers $s_\beta(x)$ and $t_\beta(x)$ are not both zero. Now suppose $\varphi, \psi \in A$ with $\varphi \neq \psi$ and $\varphi + \tau\psi = \beta + \tau\beta$. Then either $s_\varphi(x) = 0 = t_\varphi(x)$ or $s_\psi(x) = 0 = t_\psi(x)$; for otherwise $\varphi < \beta$ and $\psi < \beta$, whence $\varphi + \tau\psi < \beta + \tau\beta$. Thus

$$s_\varphi(x)s_\psi(x) = t_\varphi(x)t_\psi(x) = s_\varphi(x)t_\psi(x) = 0,$$

and our calculation above reduces to

$$\{s_\beta(x)^2 + t_\beta(x)^2\} \{2i[\tau e_\beta, e_\beta]\}_x^\dagger = 0.$$

As $\{i[\tau e_\beta, e_\beta]\}_x^\dagger \neq 0$ from $\tau\mathfrak{g}_{\beta+\tau\beta} = \mathfrak{g}_{\beta+\tau\beta}$, it follows that $s_\beta(x)^2 + t_\beta(x)^2 = 0$, which contradicts our choice of β . We conclude that $f_*D_0 \subset M_x$. That completes the proof of the theorem. Q.E.D.

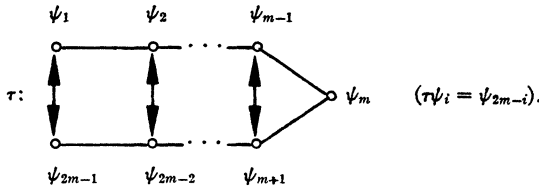
8.11. COROLLARY. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , $x \in X$, $\mathfrak{R} \subset \mathcal{O}_x$ a τ -stable Cartan subalgebra of \mathfrak{g} , and Φ a subset of a simple \mathfrak{R} -root system such that $\mathcal{O}_x = \mathfrak{O}_\Phi$. Then the following conditions are equivalent:*

- (i) $\mathfrak{N}_{[x]} = \mathfrak{Q}_{[x]}$, parabolic subalgebra defined by (8.6).
- (ii) $\mathfrak{Q}_{[x]} = \mathfrak{M}_{[x]}$, parabolic subalgebra defined by (8.8).
- (iii) If φ is an \mathfrak{R} -root such that $\langle \varphi, \delta_\Phi \rangle < 0$ and $-\varphi \notin \Phi^u \cap \tau\Phi^u$ then $\varphi + \tau\varphi$ is a root.

If those conditions hold, the holomorphic arc components of $G_0(x)$ are complex submanifolds of X .

PROOF. (iii) implies (ii) by (8.8). Given (ii), $\mathfrak{N}_{[x]}$ is an algebra, so $\mathfrak{N}_{[x]} = \mathfrak{M}_{[x]}$ by Theorem 8.9, whence (ii) says $\mathfrak{N}_{[x]} = \mathcal{Q}_{[x]}$. Given (i), $\mathcal{Q}_{[x]} \subset \mathfrak{M}_{[x]} \subset \mathfrak{N}_{[x]}$ implies (ii), and (iii) follows by (8.8). Q.E.D.

8.12. EXAMPLE. Let $G = \text{SL}(2m, \mathbb{C})$ and either $G_0 = \text{SL}(m, \mathbb{Q})$ with maximal compact subgroup $\text{Sp}(m)$, or $G_0 = \text{SL}(2m, \mathbb{R})$ with maximal compact subgroup $\text{SO}(2m)$. Choose a τ -stable Cartan subalgebra $\mathfrak{H} \subset \mathfrak{G}$ and a τ -stable simple root system Ψ . Then τ acts on Ψ by



Let $\mathcal{O} = \mathcal{O}_x = \mathcal{O}_\Phi$ where $\Phi = \{\psi_1, \dots, \psi_{2m-2}\}$. Then

$$X = G/P \text{ is complex projective } (2m - 1)\text{-space,}$$

and we note

$$\begin{aligned} \Phi^u &= \{\psi_i + \psi_{i+1} + \dots + \psi_{2m-1} : 1 \leq i \leq 2m - 1\}, \\ \tau\Phi^u &= \{\psi_1 + \psi_2 + \dots + \psi_j : 1 \leq j \leq 2m - 1\}, \end{aligned}$$

so

$$\Phi^u \cap \tau\Phi^u = \{\mu\} \text{ where } \mu = \psi_1 + \dots + \psi_{2m-1} \text{ maximal root.}$$

Thus

$$\delta_\Phi = \mu \text{ and } \mathcal{Q}_{[x]} = \mathcal{O}_\Gamma, \quad \Gamma = \{\psi_2, \psi_3, \dots, \psi_{2m-2}\}.$$

Now let B consist of all roots φ such that $\mathfrak{G}_\varphi \subset \mathcal{O}_x + \tau\mathcal{O}_x$ and $\mathfrak{G}_\varphi \not\subset \mathcal{Q}_{[x]}$. Then

$$B = \{-(\psi_1 + \dots + \psi_i), -(\psi_j + \dots + \psi_{2m-1}) : 1 \leq i < 2m - 1, 1 < j \leq 2m - 1\}.$$

If $\varphi \in B$, now $\varphi + \tau\varphi$ is of the form

$$-(\psi_1 + \dots + \psi_i) + (\psi_{2m-i} + \dots + \psi_{2m-1}),$$

which cannot be a root because we cannot have $2m - i = i + 1$. Thus

$$\text{if } \varphi \in B \text{ then } \varphi + \tau\varphi \text{ is not a root.}$$

In other words, $B = \Gamma^0$, so now

$$\mathfrak{N}_{[x]} = \mathcal{O}_x + \tau\mathcal{O}_x = \mathfrak{H} + \sum_{\varphi \in \Gamma^0} \mathfrak{G}_\varphi, \text{ not an algebra.}$$

Thus the holomorphic arc components of $G_0(x)$ are not complex submanifolds of X . In fact, as $Q_{[x]}$ is maximal among the τ -stable subalgebras of \mathfrak{g} , one expects that the $Q_{[y],0}(y)$, $y \in G_0(x)$, would be maximal among the connected complex submanifolds of X that are contained in $G_0(x)$.

The example above is based on the situation that τ preserves a simple root system Ψ but acts nontrivially on Ψ . The latter occurs if and only if τ induces an outer automorphism on the compact real form of G , i.e. if and only if the maximal compact subgroups K of G_0 have $\text{rank } K < \text{rank } G_0$. Later we will see, for the case where X is a compact hermitian symmetric space and the inclusion $G_0(x_0) \subset X$ of some open orbit is the Borel embedding of the noncompact dual of X , that every holomorphic arc component of every G_0 -orbit on X is a hermitian symmetric submanifold (the signature of metric can be indefinite there). In particular, if X is complex projective space as in the example, and if G_0 is not one of the real forms of G considered above, then it will follow that every holomorphic arc component of every G_0 -orbit is a complex submanifold of X .

8.13. LEMMA/DEFINITION. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , $x \in X$, $\mathfrak{K} \subset \mathfrak{O}_x$ a τ -stable Cartan subalgebra of \mathfrak{g} , and Φ a subset of a simple \mathfrak{K} -root system such that $\mathfrak{O}_x = \mathfrak{O}_\Phi$. Let $Q_{[x]}$ be the τ -stable parabolic subalgebra of \mathfrak{g} defined by (8.6), $Q_{[x],0}$ the real form $Q_{[x]} \cap \mathfrak{g}_0$ of $Q_{[x]}$, and $Q_{[x],0}$ the analytic subgroup of G_0 for $Q_{[x],0}$. Then*

- (i) $Q_{[x],0}(x) \subset S_{[x]}$ is a complex submanifold of X ,
- (ii) $Q_{[x]}$, $Q_{[x],0}$ and $Q_{[x],0}(x)$ are independent of choices of \mathfrak{K} and Φ ,
- (iii) if $y \in Q_{[x],0}(x)$ then $Q_{[y],0}(y) = Q_{[x],0}(x)$, and
- (iv) if $g \in G_0$ then $Q_{[gx],0}(gx) = gQ_{[x],0}(x)$.

*We will refer to the $Q_{[y],0}(y)$, $y \in G_0(x)$, as the **algebraic arc components** of $G_0(x)$.*

REMARK. Lemma 8.13 says that the algebraic arc components of $G_0(x)$ enjoy the same formal properties as the holomorphic arc components. Example 8.12 shows that they are the best substitute for the holomorphic arc components in case the latter are not complex submanifolds of X . Corollary 8.11 gives the criterion for the algebraic arc components of $G_0(x)$ to coincide with the holomorphic arc components.

PROOF. Assertion (i) is immediate from Lemma 8.4.

Suppose that $Q_{[x],0}(x)$ does not depend on the choices of \mathfrak{K} and Φ . If $g \in G_0$, then we define $Q_{[gx],0}(gx)$ from the choices $\text{ad}(g)\mathfrak{K} \subset \text{ad}(g)\mathfrak{O}_x = \mathfrak{O}_{gx}$ and $\text{ad}(g^{-1})^*\Phi$; then $Q_{[gx]} = \text{ad}(g)Q_{[x]}$, so $Q_{[gx],0} = \text{ad}(g)Q_{[x],0}$, implying the statement $Q_{[gx],0}(gx) = g \cdot Q_{[x],0}(x)$ of (iv). If $g \in Q_{[x],0}$ we obtain $Q_{[gx],0} = Q_{[x],0}$, so $Q_{[gx],0}(gx) = Q_{[x],0}(gx) = Q_{[x],0}(x)$, proving (iii). Thus we are reduced to proving (ii).

Given \mathcal{H} , $\delta_{\Phi} = \sum_{\Phi^u} \cap_{\tau\Phi^u} \varphi = \sum_{-\Phi^u} \cap_{-\tau\Phi^u} (-\psi)$ is the sum of all \mathcal{H} -roots $-\psi$ such that $\mathfrak{G}_{\psi} \subset \mathcal{O}_x + \tau\mathcal{O}_x$; thus δ_{Φ} , $Q_{[x]}$, $Q_{[x],0}$ and $Q_{[x],0}(x)$ are independent of choice of the set Φ for which $\mathcal{O}_x = \mathcal{O}_{\Phi}$, and they depend only on the choice of \mathcal{H} .

Let \mathcal{H}' and \mathcal{H}'' be two choices of \mathcal{H} . They are τ -stable Cartan subalgebras of respective reductive parts $(\mathcal{O}_x \cap \tau\mathcal{O}_x)'$ and $(\mathcal{O}_x \cap \tau\mathcal{O}_x)''$ of $\mathcal{O}_x \cap \tau\mathcal{O}_x$. Let $Q'_{[x]}$ and $Q''_{[x]}$ be the analytic subgroups of G for the corresponding algebras $\mathfrak{Q}'_{[x]}$ and $\mathfrak{Q}''_{[x]}$; let L' , L'' and L denote the respective analytic subgroups of G for $(\mathcal{O}_x \cap \tau\mathcal{O}_x)'$, $(\mathcal{O}_x \cap \tau\mathcal{O}_x)''$ and $(\mathcal{O}_x \cap \tau\mathcal{O}_x)$. Then $(L' \cup L'') \subset L \subset (Q'_{[x]} \cap Q''_{[x]})$. As any two reductive parts of L are conjugate, we have $g_1 \in L$ such that $\text{ad}(g_1)L'' = L'$; as any two Cartan subgroups of L' are conjugate, we have $g_2 \in L'$ such that $\text{ad}(g_2g_1)\mathcal{H}'' = \mathcal{H}'$. Now $g = g_2g_1$ is an element of $Q''_{[x]}$ such that $\text{ad}(g)Q'_{[x]} = Q'_{[x]}$. Thus $Q'_{[x]} = Q''_{[x]}$. It follows that $Q'_{[x],0} = Q''_{[x],0}$ and $Q'_{[x],0}(x) = Q''_{[x],0}(x)$.
 Q.E.D.

We note the relations of holomorphic and algebraic arc components with projections of flag manifolds.

8.14. PROPOSITION. *Let $P_1 \subset P_2$ be parabolic subgroups of G , $X_i = G/P_i$ the resulting complex flag manifolds, and $\pi: X_1 \rightarrow X_2$ the projection $gP_1 \rightarrow gP_2$. Let $x_2 \in X_2$ and let G_0 be a real form of G .*

1. *There exist points $x_1 \in \pi^{-1}(x_2)$ such that $G_0(x_1)$ is an open subset of $\pi^{-1}G_0(x_2)$.*

2. *If $x_1 \in \pi^{-1}(x_2)$, then the following conditions are equivalent:*

(2a) *$G_0(x_1)$ is open in $\pi^{-1}G_0(x_2)$.*

(2b) $(\mathcal{O}_1)_{x_1} + \tau(\mathcal{O}_1)_{x_1} = (\mathcal{O}_2)_{x_2} + \tau(\mathcal{O}_2)_{x_2}$.

(2c) $\text{codimension}_{\mathbb{R}}(G_0(x_1) \subset X_1) = \text{codimension}_{\mathbb{R}}(G_0(x_2) \subset X_2)$.

(2d) *Let $\mathcal{H} \subset (\mathcal{O}_1)_{x_1}$ be a τ -stable Cartan subalgebra of \mathfrak{G} , $\Phi_1 \subset \Phi_2$ subsets of a simple \mathcal{H} -root system such that $(\mathcal{O}_i)_{x_i} = \mathcal{O}_{\Phi_i}$; then $\Phi_1^u \cap \tau\Phi_1^u = \Phi_2^u \cap \tau\Phi_2^u$.*

3. *Let $x_1 \in \pi^{-1}(x_2)$ with $G_0(x_1)$ open in $\pi^{-1}G_0(x_2)$. Choose \mathcal{H} and $\Phi_1 \subset \Phi_2$ as in (2d). Then*

(3a) *the linear forms $\delta_{\Phi_1} = \delta_{\Phi_2}$, the parabolic subalgebras $Q_{[x_1]} = Q_{[x_2]}$, and the algebraic arc components of the $G_0(x_i)$ are related by*

$$Q_{[x_2],0}(x_2) = \pi Q_{[x_1],0}(x_1) \quad \text{and} \quad Q_{[x_1],0}(x_1) = G_0(x_1) \cap \pi^{-1}Q_{[x_2],0}(x_2);$$

(3b) *the spaces $\mathfrak{M}_{[x_1]} = \mathfrak{M}_{[x_2]}$, and in particular one is an algebra if and only if the other is an algebra;*

(3c) *the holomorphic arc components of $G_0(x_1)$ are complex submanifolds of X_1 if and only if the holomorphic arc components of $G_0(x_2)$ are complex submanifolds of X_2 ;*

(3d) *holomorphic arc components $S_{[x_i]} \subset G_0(x_i) \subset X_i$ are related by*

$$S_{[x_2]} = \pi S_{[x_1]} \text{ and } S_{[x_1]} = G_0(x_1) \cap \pi^{-1}S_{[x_2]};$$

(3e) the algebras $\mathfrak{N}_{[x_1]} = \mathfrak{N}_{[x_2]}$ and the groups $N_{[x_1],0} = N_{[x_2],0}$.

PROOF. The projection $\pi: X_1 \rightarrow X_2$ is G_0 -invariant, so $\pi^{-1}G_0(x_2)$ is a union of G_0 -orbits. As there are only finitely many G_0 -orbits on X_1 , now $\pi^{-1}G_0(x_2)$ is a finite union of G_0 -orbits, so at least one of them has full dimension in $\pi^{-1}G_0(x_2)$. That proves (1).

If $G_0(x_1)$ is open in $\pi^{-1}G_0(x_2)$, then

$$\begin{aligned} \text{codimension}_R(G_0(x_1) \subset X_1) &= \dim_R X_1 - \dim_R G_0(x_1) \\ &= \dim_R X_1 - \dim_R \pi^{-1}G_0(x_2) \\ &= [\dim_R X_2 + \{\dim_R X_1 - \dim_R X_2\}] \\ &\quad - [\dim_R G_0(x_2) + \{\dim_R X_1 - \dim_R X_2\}] \\ &= \dim_R X_2 - \dim_R G_0(x_2) \\ &= \text{codimension}_R(G_0(x_2) \subset X_2). \end{aligned}$$

Conversely, if those codimensions are equal, the argument shows that $\dim_R G_0(x_1) = \dim_R \pi^{-1}G_0(x_2)$, so $G_0(x_1)$ is open in $\pi^{-1}G_0(x_2)$. Thus (2a) and (2c) are equivalent. Theorem 2.12 (ii) says that (2c) is equivalent to $|\Phi_1^u \cap \tau \Phi_1^u| = |\Phi_2^u \cap \tau \Phi_2^u|$. As $\Phi_2^u \subset \Phi_1^u$, the latter is equivalent to $\Phi_1^u \cap \tau \Phi_1^u = \Phi_2^u \cap \tau \Phi_2^u$. Thus (2c) and (2d) are equivalent. We reformulate (2d) as $-\Phi_1^u \cap -\tau \Phi_1^u = -\Phi_2^u \cap -\tau \Phi_2^u$, i.e. as $(\mathcal{P}_1)_{x_1}^{-u} \cap \tau(\mathcal{P}_1)_{x_1}^{-u} = (\mathcal{P}_2)_{x_2}^{-u} \cap \tau(\mathcal{P}_2)_{x_2}^{-u}$, i.e. as $(\mathcal{P}_1)_{x_1} + \tau(\mathcal{P}_1)_{x_1} = (\mathcal{P}_2)_{x_2} + \tau(\mathcal{P}_2)_{x_2}$, which is (2b). That completes the proof of (2).

Let $x_1 \in \pi^{-1}(x_2)$ with $G_0(x_1)$ open in $\pi^{-1}G_0(x_2)$. Then we have (2b), (2c) and (2d). From (2d), $\delta_{\Phi_1} = \delta_{\Phi_2}$, so $\mathcal{Q}_{[x_1]} = \mathcal{Q}_{[x_2]}$, $\mathcal{Q}_{[x_1],0} = \mathcal{Q}_{[x_2],0}$ and $\mathcal{Q}_{[x_2],0}(x_2) = \pi \mathcal{Q}_{[x_1],0}(x_1)$; and $\mathcal{O}_{x_1} \cap \mathcal{G}_0 \subset \mathcal{Q}_{[x_1],0}$ implies further that $\mathcal{Q}_{[x_1],0}(x_1) = G_0(x_1) \cap \pi^{-1}\mathcal{Q}_{[x_2],0}(x_2)$, completing the proof of (3a). $\mathcal{Q}_{[x_1]} = \mathcal{Q}_{[x_2]}$ together with (2b) implies $\mathfrak{M}_{[x_1]} = \mathfrak{M}_{[x_2]}$, proving (3b). Theorem 8.9 and (3b) imply (3c). If the holomorphic arc components are complex manifolds then (3d) follows from (3b) and Theorem 8.9. But in general we must look at holomorphic arcs.

If $f: D \rightarrow X_1$ is a holomorphic arc in $G_0(x_1)$ then $(\pi \cdot f): D \rightarrow X_2$ is a holomorphic arc in $\pi G_0(x_1) = G_0(x_2)$. Thus $\pi S_{[x_1]} \subset S_{[x_2]}$. Let $S = G_0(x_1) \cap \pi^{-1}S_{[x_2]}$, $p = \pi|_S$, and ξ the fibration $p: S \rightarrow S_{[x_2]}$. Then ξ is locally trivial and its fibres carry complex manifold structures as open subsets of π -fibres. If $y \in S$ this implies $S_{[y]} = p^{-1}pS_{[y]}$. Thus $S_{[x_2]}$ is a disjoint union of sets $pS_{[y]} = \pi S_{[y]}$, $y \in S$, and S is disjoint union of the corresponding $p^{-1}\pi S_{[y]}$. As the images of holomorphic arcs in $G_0(x_2)$ fill out a neighborhood in $S_{[x_2]}$ of every point $p(y) \in S_{[x_2]}$, and as those holomorphic arcs lift to S , the sets $p^{-1}\pi S_{[y]}$ are open in S , so

$S = p^{-1}\pi S_{[x_1]}$ by connectedness. That proves (3d). It follows that $\mathfrak{N}_{[x_1],0} \equiv \mathfrak{N}_{[x_2],0}$ modulo the isotropy subalgebras of \mathfrak{G}_0 at x_1 and x_2 . As those isotropy subalgebras $(\mathcal{P}_i)_{x_i} \cap \mathfrak{G}_0 = (\mathcal{P}_i)_{x_i} \cap \tau(\mathcal{P}_i)_{x_i} \cap \mathfrak{G}_0 \subset \mathcal{Q}_{[x_i]} \cap \mathfrak{G}_0 \subset \mathfrak{N}_{[x_i],0}$, now $\mathfrak{N}_{[x_1],0} = \mathfrak{N}_{[x_2],0}$; (3e) follows. Q.E.D.

We describe the "space of holomorphic arc components" of a G_0 -orbit on X .

8.15. THEOREM. *Let $X = G/P$ be a complex flag manifold, $x \in X$, G_0 a real form of G , and K a maximal compact subgroup of G_0 . Define projections*

$$(8.16a) \quad \sigma: G_0(x) \rightarrow G_0/(G_0 \cap N_{[x]}) \quad \text{by } \sigma(gx) = g(G_0 \cap N_{[x]}),$$

$$(8.16b) \quad \kappa: G_0(x) \rightarrow G_0/(G_0 \cap Q_{[x]}) \quad \text{by } \kappa(gx) = g(G_0 \cap Q_{[x]}),$$

and

$$(8.16c) \quad \nu: G_0/(G_0 \cap Q_{[x]}) \rightarrow G_0/(G_0 \cap N_{[x]})$$

$$\text{by } g(G_0 \cap Q_{[x]}) \rightarrow g(G_0 \cap N_{[x]}).$$

1. σ is well defined and (8.16a) is a G_0 -equivariant differentiable fibre bundle with

(1a) structure group $G_0 \cap N_{[x]}$, the parabolic subgroup of G_0 with identity component $N_{[x],0}$,

(1b) fibre over $\sigma(gx)$ equal to the holomorphic arc component $S_{[gx]} = gS_{[x]}$ of $G_0(x)$ through $g(x)$, and

(1c) compact base $G_0/(G_0 \cap N_{[x]}) = K/(K \cap N_{[x]})$, real flag manifold that parameterizes the holomorphic arc components of $G_0(x)$.

2. κ is well defined and (8.16b) is a G_0 -equivariant differentiable fibre bundle with

(2a) structure group $G_0 \cap Q_{[x]}$, the parabolic subgroup of G_0 with identity component $Q_{[x],0}$,

(2b) fibre over $\kappa(gx)$ equal to the algebraic arc component $Q_{[gx],0}(gx) = gQ_{[x],0}(x)$ of $G_0(x)$ through $g(x)$, and

(2c) compact base $G_0/(G_0 \cap Q_{[x]}) = K/(K \cap Q_{[x]})$, real flag manifold that parameterizes the algebraic arc components of $G_0(x)$.

3. $\nu = \sigma \cdot \kappa^{-1}$, well-defined projection of real flag manifolds, and (8.16c) has fibre over $\nu(gx)$ parameterizing the algebraic arc components of $G_0(x)$ that are contained in the holomorphic arc component $S_{[gx]}$.

PROOF. $\mathcal{P}_x \cap \tau \mathcal{P}_x \subset \mathcal{Q}_{[x]} \subset \mathfrak{N}_{[x]}$ implies $G_0 \cap P_x \subset G_0 \cap Q_{[x]} \subset G_0 \cap N_{[x]}$, so σ, κ and $\nu = \sigma \cdot \kappa^{-1}$ are well defined, and (8.16a, b, c) are G_0 -equivariant differentiable fibre bundles with structure groups as described in (1a) and (2a).

Let F be the complex flag manifold $G/N_{[x]}$ or $G/Q_{[x]}$. View G as a

linear algebraic group def/\mathbb{R} such that G_0 is the topological identity component of the set $G_{\mathbb{R}}$ of real points. Then F is def/\mathbb{R} and Theorem 3.6(2) says that the G_0 -orbit of the identity coset on F is $F_{\mathbb{R}}$, and that it is the unique closed G_0 -orbit on F . That proves that the base spaces of (8.16a) and (8.16b) are real flag manifolds and that K acts transitively on them. Now it only remains to prove (1b) and (2b); for then (1c), (2c) and (3c) will follow immediately.

Choose a τ -stable Cartan subalgebra $\mathfrak{H} \subset \mathfrak{P}_{\mathfrak{z}}$ of \mathfrak{g} and decompose $\mathfrak{H}_0 = \mathfrak{H} \cap \mathfrak{G}_0$ as $\mathfrak{H}_T + \mathfrak{H}_V$, toroidal and vector parts relative to a Cartan involution θ of \mathfrak{G}_0 . The choice of \mathfrak{H} determines reductive parts $\mathfrak{P}'_{\mathfrak{z}}$, $\mathfrak{Q}'_{[\mathfrak{z}]}$ and $\mathfrak{N}'_{[\mathfrak{z}]}$; let $\mathfrak{Z}_P \subset \mathfrak{H}$, $\mathfrak{Z}_Q \subset \mathfrak{H}$ and $\mathfrak{Z}_N \subset \mathfrak{H}$ denote their respective centers, and let $\mathfrak{Z} \subset \mathfrak{H}$ denote the center of $\mathfrak{P}'_{\mathfrak{z}} \cap \tau \mathfrak{P}'_{\mathfrak{z}}$; note $\tau \mathfrak{Z} = \mathfrak{Z}$, $\tau \mathfrak{Z}_Q = \mathfrak{Z}_Q$, $\tau \mathfrak{Z}_N = \mathfrak{Z}_N$ and $\mathfrak{Z}_N \subset \mathfrak{Z}_Q \subset \mathfrak{Z}$.

$\mathfrak{N}'_{[\mathfrak{z}]}$ is the centralizer of \mathfrak{Z}_N in \mathfrak{g} , $\mathfrak{Q}'_{[\mathfrak{z}]}$ is the centralizer of \mathfrak{Z}_Q in \mathfrak{g} , and $\mathfrak{P}'_{\mathfrak{z}}$ is the centralizer of \mathfrak{Z}_P in \mathfrak{g} , because $\mathfrak{N}_{[\mathfrak{z}]}$, $\mathfrak{Q}_{[\mathfrak{z}]}$ and $\mathfrak{P}_{\mathfrak{z}}$ are parabolic subalgebras of \mathfrak{g} . Now $\mathfrak{P}'_{\mathfrak{z}} \cap \tau \mathfrak{P}'_{\mathfrak{z}}$ is the centralizer of $\mathfrak{Z}_P + \tau \mathfrak{Z}_P$ in \mathfrak{g} , and it follows that $\mathfrak{P}'_{\mathfrak{z}} \cap \tau \mathfrak{P}'_{\mathfrak{z}}$ is the centralizer of \mathfrak{Z} in \mathfrak{g} . Decompose real forms:

$$\begin{aligned} \mathfrak{Z}_{N,0} &= \mathfrak{Z}_N \cap \mathfrak{G}_0 & \text{and} & & \mathfrak{Z}_{N,0} &= \mathfrak{Z}_{N,T} + \mathfrak{Z}_{N,V}, \\ \mathfrak{Z}_{Q,0} &= \mathfrak{Z}_Q \cap \mathfrak{G}_0 & \text{and} & & \mathfrak{Z}_{Q,0} &= \mathfrak{Z}_{Q,T} + \mathfrak{Z}_{Q,V}, \\ \mathfrak{Z}_0 &= \mathfrak{Z} \cap \mathfrak{G}_0 & \text{and} & & \mathfrak{Z}_0 &= \mathfrak{Z}_T + \mathfrak{Z}_V \end{aligned}$$

into toroidal and vector parts by intersecting with \mathfrak{H}_T and \mathfrak{H}_V . If $\mathfrak{s} \subset \mathfrak{g}$ then $Z(\mathfrak{s})$ denotes the centralizer of \mathfrak{s} in G and $Z_0(\mathfrak{s})$ denotes the centralizer $G_0 \cap Z(\mathfrak{s})$ of \mathfrak{s} in G_0 . Now we have $N'_{[\mathfrak{z}]} = Z(\mathfrak{Z}_{N,T}) \cap Z(\mathfrak{Z}_{N,V})$, $Q'_{[\mathfrak{z}]} = Z(\mathfrak{Z}_{Q,T}) \cap Z(\mathfrak{Z}_{Q,V})$ and $P'_z \cap \tau P'_z = Z(\mathfrak{Z}_T) \cap Z(\mathfrak{Z}_V)$; for $Z(\mathfrak{s}) = Z(\mathfrak{s} \cap \mathfrak{G}_0)$ whenever \mathfrak{s} is a τ -stable subspace of \mathfrak{g} . Thus

$$\begin{aligned} G_0 \cap N'^r_{[\mathfrak{z}]} &= Z_0(\mathfrak{Z}_{N,T}) \cap Z_0(\mathfrak{Z}_{N,V}), \\ G_0 \cap Q'^r_{[\mathfrak{z}]} &= Z_0(\mathfrak{Z}_{Q,T}) \cap Z_0(\mathfrak{Z}_{Q,V}), \end{aligned}$$

and

$$G_0 \cap P'^r_z \cap \tau P'^r_z = Z_0(\mathfrak{Z}_T) \cap Z_0(\mathfrak{Z}_V).$$

It follows that

$$G_0 \cap \exp(i\mathfrak{Z}_{N,V}) \text{ meets every component of } G_0 \cap N'^r_{[\mathfrak{z}]},$$

$$G_0 \cap \exp(i\mathfrak{Z}_{Q,V}) \text{ meets every component of } G_0 \cap Q'^r_{[\mathfrak{z}]},$$

and

$$G_0 \cap \exp(i\mathfrak{Z}_V) \text{ meets every component of } G_0 \cap P'^r_z \cap \tau P'^r_z.$$

Using $(\mathcal{O}_x \cap \tau \mathcal{O}_x)^r = \mathcal{O}'_x \cap \tau \mathcal{O}'_x$, we also have

$$G_0 \cap \exp(iZ_V) \text{ meets every component of } G_0 \cap (P_x \cap \tau P_x)^r.$$

Connectivity of real unipotent groups says that $G_0 \cap N'_{[x]}$ meets every component of $G_0 \cap N_{[x]}$, $G_0 \cap Q'_{[x]}$ meets every component of $G_0 \cap Q_{[x]}$, and $G_0 \cap (P_x \cap \tau P_x)^r$ meets every component of $G_0 \cap (P_x \cap \tau P_x) = G_0 \cap P_x$. Thus

$$G_0 \cap \exp(iZ_{N,V}) \text{ meets every component of } G_0 \cap N_{[x]},$$

$$G_0 \cap \exp(iZ_{Q,V}) \text{ meets every component of } G_0 \cap Q_{[x]},$$

and

$$G_0 \cap \exp(iZ_V) \text{ meets every component of } G_0 \cap P_x.$$

$Z_N \subset Z_Q \subset Z$ implies $iZ_{N,V} \subset iZ_{Q,V} \subset iZ_V$, so $G_0 \cap \exp(iZ_{N,V}) \subset G_0 \cap \exp(iZ_{Q,V}) \subset G_0 \cap \exp(iZ_V)$. Thus $G_0 \cap P_x$ meets every component of $G_0 \cap Q_{[x]}$ and also meets every component of $G_0 \cap N_{[x]}$. This proves that $(G_0 \cap Q_{[x]})(x)$ and $(G_0 \cap N_{[x]})(x)$ are connected, hence equal to $Q_{[x],0}(x)$ and $N_{[x],0}(x) = S_{[x]}$, respectively. That proves (1b) and (2b), completing the proof of the theorem. Q.E.D.

8.17. DEFINITIONS. *In the notation of Theorem 8.15, the base space $G_0/(G_0 \cap N_{[x]})$ of the fibre bundle (8.16a) will be called the **space of holomorphic arc components** of $G_0(x)$, and the base $G_0/(G_0 \cap Q_{[x]})$ of the bundle (8.16b) will be called the **space of algebraic arc components** of $G_0(x)$. This terminology is based on results (1c) and (2c) of Theorem 8.15.*

8.18. COROLLARY. *Let $P_1 \subset P_2$ be parabolic subgroups of G , $X_i = G/P_i$ the resulting complex flag manifolds, $\pi: X_1 \rightarrow X_2$ the projection $gP_1 \mapsto gP_2$, G_0 a real form of G , $x_2 \in X_2$, and $x_1 \in \pi^{-1}(x_2)$ such that $G_0(x_1)$ is open in $\pi^{-1}G_0(x_2)$.*

1. *π induces a diffeomorphism $S_{[x_1]} \mapsto \pi S_{[x_1]} = S_{[\pi x_1]}$ from the space of holomorphic arc components of $G_0(x_1)$ onto the space of holomorphic arc components of $G_0(x_2)$.*

2. *π induces a diffeomorphism $Q_{[x_1],0}(x) \mapsto \pi Q_{[x_1],0}(x) = Q_{[\pi x_1],0}(\pi x)$ from the space of algebraic arc components of $G_0(x_1)$ onto the space of algebraic arc components of $G_0(x_2)$.*

PROOF. This is immediate from Theorem 8.15 and the results $Q_{[x_1]} = Q_{[x_2]}$ and $N_{[x_1]} = N_{[x_2]}$ of Proposition 8.14(3). Q.E.D.

8.19. COROLLARY. *Let $X = G/P$ be a complex flag manifold, $x \in X$, and G_0 a real form of G . Then the following conditions are equivalent:*

- (i) $G_0(x)$ is the unique closed G_0 -orbit on X .
- (ii) $G_0(x)$ is compact.

- (iii) *The holomorphic arc components of $G_0(x)$ are compact.*
- (iv) *The algebraic arc components of $G_0(x)$ are compact.*

PROOF. We have noted equivalence of (i) and (ii) from compactness of X . Now consider the fibrations (8.16)

$$\begin{aligned} \sigma: G_0(x) &\rightarrow G_0/(G_0 \cap N_{[x]}), \\ \kappa: G_0(x) &\rightarrow G_0/(G_0 \cap Q_{[x]}) \end{aligned}$$

whose base spaces, both compact, are the respective spaces of holomorphic and algebraic arc components of $G_0(x)$. As the base is compact, the total space is compact if and only if the fibres are compact. That given equivalence of (ii) with (iii) and with (iv). Q.E.D.

8.20. COROLLARY. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , and $x \in X$. If the holomorphic arc components of $G_0(x)$ are points, then $G_0(x)$ is the closed G_0 -orbit on X . Semiconversely, if $G_0(x)$ is the closed G_0 -orbit on X , and if $\dim_{\mathbb{R}} G_0(x) = \dim_{\mathbb{C}} X$, then the holomorphic arc components of $G_0(x)$ are points.*

PROOF. The first assertion follows from Corollary 8.19 because points are compact. Now suppose $\dim_{\mathbb{R}} G_0(x) = \dim_{\mathbb{C}} X$. Then $\mathcal{O}_x = \tau\mathcal{P}_x$ by Theorem 3.6, so $\mathcal{O}_x \cap \tau\mathcal{P}_x = \mathcal{O}_x + \tau\mathcal{P}_x$, and the construction (8.8) yields $\mathfrak{N}_{[x]}$ as the algebra \mathcal{O}_x . Theorem 8.9 then says $\mathfrak{N}_{[x]} = \mathcal{P}_x$, so $S_{[x]} = N_{[x],0}(x) = (G_0 \cap P_x)(x) = x$. Q.E.D.

9. **Global conditions for the components of an orbit.** We will study certain conditions related to complexity and measurability for the holomorphic arc components of a G_0 -orbit. Those conditions are defined as follows.

9.1. DEFINITIONS. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , $x \in X$, $S_{[x]}$ the holomorphic arc component of $G_0(x)$ through x , $N_{[x],0}$ the identity component of the normalizer of $S_{[x]}$ in G_0 , and $N_{[x]} = N_{[x],0}^C$.*

1. *The orbit $G_0(x)$ is called **partially complex** if its holomorphic arc components are complex submanifolds of X , i.e. if $S_{[x]}$ is a complex submanifold of X .*

2. *The orbit $G_0(x)$ is said to be of **flag type** if the $N_{[x']}(x')$, $x' \in G_0(x)$, are complex flag manifolds, i.e. if $N_{[x]}(x)$ is a complex flag manifold, i.e. if $N_{[x]}$ has a τ -stable reductive part $N_{[x]}^r$ that is transitive on $N_{[x]}(x)$ and that has $N_{[x]}^r \cap P_x$ as a parabolic subgroup.*

3. *The orbit $G_0(x)$ is called **measurable** if its holomorphic arc components carry invariant positive Radon measures, i.e. if $S_{[x]}$ carries an $N_{[x],0}$ -invariant positive Radon measure.*

4. The orbit $G_0(x)$ is said to be **polarized** if the $\mathcal{O}_{x'}$, $x' \in G_0(x)$, have τ -stable reductive parts, i.e. if $\tau\mathcal{O}'_x = \mathcal{O}'_x$ for some choice of reductive part $\mathcal{O}'_x \subset \mathcal{O}_x$.

5. The orbit $G_0(x)$ is called **integrable** if the distribution T of (8.10) is integrable, i.e. if $\mathcal{O}_x + \tau\mathcal{O}_x$ is a subalgebra of \mathfrak{g} .

Theorem 8.9 gives several conditions for $G_0(x)$ to be partially complex. The analogous condition for algebraic arc components is automatic (Lemma 8.13), so it will not be given a name.

The notion of flag type is a useful hereditary condition. The analogous condition for algebraic arc components is that $Q_{[x]}(x)$ be a complex flag manifold. In that case we will say that $G_0(x)$ is of **flag type relative to its algebraic arc components**.

The notion of measurable orbit is the primary object of this paper. In §6 we obtained the basic facts for measurable open orbits, and the definition here specializes to that of §6. The analogous condition for algebraic arc components is that $Q_{[x],0}(x)$ carry a $Q_{[x],0}$ -invariant positive Radon measure. In that case we will say that $G_0(x)$ is **measurable relative to its algebraic arc components**.

The notion of polarized orbit will be useful in studying the three concepts above. Let $\mathcal{K} \subset \mathcal{O}_x$ be any τ -stable Cartan subalgebra of \mathfrak{g} . That determines a choice of \mathcal{O}'_x , and Lemma 2.10 says $(\mathcal{O}_x \cap \tau\mathcal{O}_x)^r = \mathcal{O}'_x \cap \tau\mathcal{O}'_x$. Thus $G_0(x)$ is polarized if and only if $\dim(\mathcal{O}_x \cap \tau\mathcal{O}_x)^r = \dim \mathcal{O}'_x$. The latter can be expressed as $\dim \{(\mathcal{O}_x \cap \tau\mathcal{O}_x)/(\mathcal{O}_x \cap \tau\mathcal{O}_x)^u\} = \dim \{\mathcal{O}_x/\mathcal{O}_x^u\}$, which is independent of choice of \mathcal{K} . Thus $G_0(x)$ is polarized if and only if $\mathcal{O}'_x = \tau\mathcal{O}'_x$ relative to every τ -stable Cartan subalgebra $\mathcal{K} \subset \mathcal{O}_x$ of \mathfrak{g} .

The notion of integrable orbit is recalled from §7, where it was studied in some detail.

9.2. THEOREM. Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , $x \in X$, $\mathcal{K} \subset \mathcal{O}_x$ a τ -stable Cartan subalgebra of \mathfrak{g} , and Φ a subset of a simple \mathcal{K} -root system such that $\mathcal{O}_x = \mathcal{O}_\Phi$. Define

$$(9.3) \quad \mathfrak{U}_x^+ = \sum_{\Phi^u \cap -\tau\Phi^u} \mathfrak{g}_\varphi, \quad \mathfrak{U}_x^- = \tau\mathfrak{U}_x^+ = \sum_{-\Phi^u \cap \tau\Phi^u} \mathfrak{g}_\varphi, \quad \mathfrak{U}_x = \mathfrak{U}_x^+ + \mathfrak{U}_x^-.$$

Then $G_0(x)$ is measurable if and only if

$$(9.4) \quad \mathfrak{N}_{[x]}^u = (\mathcal{O}_x \cap \tau\mathcal{O}_x)^u, \quad \mathfrak{N}_{[x]}^r = (\mathcal{O}_x \cap \tau\mathcal{O}_x)^r + \mathfrak{U}_x, \\ \mathfrak{N}_{[x]} = (\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x.$$

If $G_0(x)$ is measurable, then

- (i) $G_0(x)$ is partially complex;
- (ii) $G_0(x)$ is of flag type;

- (iii) $G_0(x)$ is polarized if and only if it is integrable; and
- (iv) the invariant Radon measure on $S_{[x]}$ is the volume element of an $N_{[x],0}$ -invariant kaehler metric (nondegenerate, but not necessarily positive definite).

PROOF. Let A be the set of \mathcal{H} -roots such that $\mathfrak{N}_{[x]} = (\mathcal{O}_x \cap \tau \mathcal{O}_x) + \sum_A \mathcal{G}_\varphi$ and define $\alpha = \sum_A \varphi$. If $G_0(x)$ is measurable, then $S_{[x]}$ has a positive $N_{[x],0}$ -invariant Radon measure μ . Then the coset space $N_{[x],0}/(N_{[x],0} \cap P_x) = N_{[x],0}/(P_x \cap \tau P_x)_0$ representing $S_{[x]}$ has μ as invariant measure, so $(P_x \cap P_x)_0$ acts with determinant ± 1 on its tangent space, i.e. $(\mathcal{O}_x \cap \tau \mathcal{O}_x)_0$ acts with trace 0 on $\mathfrak{N}_{[x],0}/(\mathcal{O}_x \cap \tau \mathcal{O}_x)_0$, i.e. $\mathcal{O}_x \cap \tau \mathcal{O}_x$ acts with trace 0 on $\mathfrak{N}_{[x]}/(\mathcal{O}_x \cap \tau \mathcal{O}_x)$, i.e. $(\mathcal{O}_x \cap \tau \mathcal{O}_x)^r$ acts with trace 0 on $\sum_A \mathcal{G}_\varphi$, i.e. \mathcal{H} acts with trace 0 on $\sum_A \mathcal{G}_\varphi$, i.e. $\alpha = 0$. The argument is reversible. Thus $G_0(x)$ is measurable if and only if $\alpha = 0$.

Suppose $G_0(x)$ measurable. If $\varphi \in A$ with $\mathcal{G}_\varphi \subset \mathfrak{N}_{[x]}^\mu$, then α has positive inner product with the sum of the roots of $\mathfrak{N}_{[x]}^\mu$, contradicting $\alpha = 0$. Thus $\sum_A \mathcal{G}_\varphi \subset \mathfrak{N}_{[x]}^\mu$ and it follows that $\mathfrak{N}_{[x]}^\mu \subset (\mathcal{O}_x \cap \tau \mathcal{O}_x)$. Now $A = -A$ by Lemma 6.2, for $\mathfrak{N}_{[x]}^\mu \cap (\mathcal{O}_x \cap \tau \mathcal{O}_x)$ is reductive. As A is disjoint from $\Phi^* \cap \tau \Phi^*$, i.e. as

$$A \subset (\Phi^r \cap -\tau \Phi^\mu) \cup (-\Phi^\mu \cap \tau \Phi^r) \cup (\Phi^\mu \cap -\tau \Phi^\mu) \cup (-\Phi^\mu \cap \tau \Phi^\mu),$$

this says $\sum_A \mathcal{G}_\varphi \subset \mathcal{U}_x$. As $\mathfrak{N}_{[x]}$ is parabolic now $\sum_A \mathcal{G}_\varphi = \mathcal{U}_x$. Thus $\mathfrak{N}_{[x]} = (\mathcal{O}_x \cap \tau \mathcal{O}_x) + \mathcal{U}_x$. That proves (9.4).

Conversely assume (9.4). Then $N_{[x],0}^\mu$ acts trivially on $S_{[x]}$ and $N_{[x],0}^r$ acts transitively with reductive isotropy subgroup $(P_x^r \cap \tau P_x^r)_0$, so Lemma 6.2 says that $G_0(x)$ is measurable.

Suppose $G_0(x)$ measurable. Then (9.4) implies $\mathfrak{N}_{[x]} \subset (\mathcal{O}_x + \tau \mathcal{O}_x)$, so Theorem 8.9 says that $G_0(x)$ is partially complex; that proves (i). $\mathfrak{N}_{[x]}^\mu \subset \mathcal{O}_x$ by (9.4), so $N_{[x]}^\mu$ acts trivially on $N_{[x]}(x)$, whence $N_{[x]}^r$ is transitive on $N_{[x]}(x)$; the isotropy subalgebra of $\mathfrak{N}_{[x]}^r$ at x is $\mathfrak{N}_{[x]}^r \cap \mathcal{O}_x = (\mathcal{O}_x \cap \tau \mathcal{O}_x)^r + \mathcal{U}_x^+$, parabolic subalgebra; thus $N_{[x]}(x)$ is a complex flag manifold; that proves (ii). If $G_0(x)$ is polarized then $\tau \Phi^r = \Phi^r$, so $(\mathcal{O}_x \cap \tau \mathcal{O}_x) = \mathcal{O}_x^r + \tau \mathcal{O}_x^r + (\mathcal{O}_x^\mu \cap \tau \mathcal{O}_x^\mu)$ so $(\mathcal{O}_x \cap \tau \mathcal{O}_x) + \mathcal{U}_x = (\mathcal{O}_x + \tau \mathcal{O}_x)$, and (9.4) says that $\mathfrak{N}_{[x]} = (\mathcal{O}_x + \tau \mathcal{O}_x)$; thus $\mathcal{O}_x + \tau \mathcal{O}_x$ is an algebra and $G_0(x)$ is integrable. If $G_0(x)$ is integrable then $\mathfrak{N}_{[x]} = (\mathcal{O}_x + \tau \mathcal{O}_x)$ by Theorem 8.9, for $\mathcal{Q}_{[x]} = (\mathcal{O}_x \cap \tau \mathcal{O}_x)$; then (9.4) says that $\Phi^r \cap -\tau \Phi^\mu$ and $-\Phi^\mu \cap \tau \Phi^r$ are empty, so $\tau \Phi^r = \Phi^r$ and $G_0(x)$ is polarized. That proves (iii).

Suppose $G_0(x)$ measurable. By (ii), $Y = N_{[x]}(x)$ is a complex flag manifold $N_{[x]}^r/(N_{[x]}^r \cap P_x)$ on which $N_{[x]}^\mu$ acts trivially. Lemma 8.4 says that $S_{[x]} = N_{[x],0}^r(x)$ is an open $N_{[x],0}^r$ -orbit on Y . Now $S_{[x]}$ is a measurable open $N_{[x],0}^r$ -orbit on the complex flag manifold Y , so

Theorem 6.3 says that its measure is the volume element of a (possibly indefinite) $N_{[x],0}^r$ -invariant kaehler metric ds^2 . As $N_{[x],0}^u$ acts trivially on $S_{[x]}$ it must preserve ds^2 . Thus ds^2 is $N_{[x],0}$ -invariant. That proves (iv), completing the proof of the theorem. Q.E.D.

9.5. COROLLARY. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , $x \in X$, and \mathcal{U}_x as defined in (9.3). Then $G_0(x)$ is measurable relative to its algebraic arc components if and only if*

$$(9.6) \quad \begin{aligned} \mathcal{Q}_{[x]}^u &= (\mathcal{P}_x \cap \tau\mathcal{P}_x)^u, & \mathcal{Q}_{[x]}^r &= (\mathcal{P}_x \cap \tau\mathcal{P}_x)^r + \mathcal{U}_x, \\ \mathcal{Q}_{[x]} &= (\mathcal{P}_x \cap \tau\mathcal{P}_x) + \mathcal{U}_x. \end{aligned}$$

In that case,

- (i) $G_0(x)$ is of flag type relative to its algebraic arc components;
- (ii) $G_0(x)$ is polarized if and only if it is integrable;
- (iii) the invariant Radon measure on $Q_{[x],0}(x)$ is the volume element of a $Q_{[x],0}$ -invariant, possibly indefinite, kaehler metric.

PROOF. $\mathcal{Q}_{[x]}$ can be substituted for $\mathfrak{N}_{[x]}$ throughout the proof of Theorem 9.2. Q.E.D.

To decide whether a given $G_0(x)$ is measurable, compute δ_Φ and $\mathcal{Q}_{[x]}$ by (8.6), then $\mathfrak{M}_{[x]}$ by (8.8), and finally $\mathfrak{W}_{[x]} = (\mathcal{P}_x \cap \tau\mathcal{P}_x) + \mathcal{U}_x$ by (9.3). Then $G_0(x)$ is measurable if and only if (i) $\mathfrak{M}_{[x]} = \mathfrak{W}_{[x]}$ and (ii) $\mathfrak{W}_{[x]}$ is an algebra. It is easy to check (i), and it follows from (9.3) that (ii) is equivalent to $[\mathcal{U}_x^+, \mathcal{U}_x^-] \subset \mathfrak{W}_{[x]}$. The latter inclusion can be unpleasant to check, but it is much more easily decidable than is the inclusion $[\mathfrak{M}_{[x]}, \mathfrak{M}_{[x]}] \subset \mathfrak{M}_{[x]}$ that says $\mathfrak{M}_{[x]}$ is an algebra. So measurability is easier to check than is partial complexity.

9.7. THEOREM. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , $x \in X$, $\mathfrak{K} \subset \mathcal{P}_x$ a τ -stable Cartan subalgebra of \mathfrak{G} , Φ a subset of a simple \mathfrak{K} -root system such that $\mathcal{P}_x = \mathcal{P}_\Phi$, and \mathcal{U}_x as in (9.3).*

1. $G_0(x)$ is of flag type if and only if $\mathcal{U}_x \subset \mathfrak{N}_{[x]}$.
2. If $G_0(x)$ is of flag type, then $G_0(x)$ is partially complex if and only if $S_{[x]} = N_{[x],0}^r(x)$ is an open $N_{[x],0}^r$ -orbit on the complex flag manifold $N_{[x]}^r(x)$.
3. If $G_0(x)$ is partially complex and of flag type, then $\dim_{\mathbb{C}} S_{[x]} \cong |\Phi^u \cap -\tau\Phi^u|$, with equality if and only if $G_0(x)$ is measurable.

PROOF. Suppose $G_0(x)$ of flag type, so $N_{[x]}^r$ is transitive on $Y = N_{[x]}^r(x)$ and $N_{[x]}^r \cap P_x$ is parabolic in $N_{[x]}^r$. As $N_{[x]}^u$ is solvable it has a fixed point $y \in Y$. If $g \in N_{[x]}^r$ then $N_{[x]}^u(gy) = gN_{[x]}^u(y) = gy$; thus $N_{[x]}^u$ acts trivially on Y ; now $\mathfrak{N}_{[x]}^u \subset \mathcal{P}_x$, so $\tau\mathfrak{N}_{[x]}^u = \mathfrak{N}_{[x]}^u$ gives us $\mathfrak{N}_{[x]}^u \subset (\mathcal{P}_x \cap \tau\mathcal{P}_x)$. If $\varphi \in \Phi^u \cap -\tau\Phi^u$, it follows that neither φ nor $-\varphi$ is a root of $\mathfrak{N}_{[x]}^u$, so both are roots of $\mathfrak{N}_{[x]}^r$; thus $\mathcal{U}_x \subset \mathfrak{N}_{[x]}$.

Conversely suppose $\mathfrak{U}_x \subset \mathfrak{N}_{[x]}$. Then $\mathfrak{N}_{[x]}^r$ contains $(\mathcal{O}_x^r + \tau\mathcal{O}_x^r) + \mathfrak{U}_x$, and $\mathfrak{N}_{[x]}$ contains $(\mathcal{O}_x \cap \tau\mathcal{O}_x)^u$; it follows that $\mathfrak{N}_{[x]}^u \subset (\mathcal{O}_x \cap \tau\mathcal{O}_x)^u$, so $N_{[x]}^r$ is transitive on $N_{[x]}(x)$. The isotropy subalgebra $\mathfrak{N}_{[x]}^r \cap \mathcal{O}_x$ contains $(\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x^+$, so it is parabolic in $\mathfrak{N}_{[x]}^r$. Thus $G_0(x)$ is of flag type.

Let $G_0(x)$ be of flag type, $Y = N_{[x]}(x) = N_{[x]}^r(x)$. Then $S_{[x]} = N_{[x],0}(x) = N_{[x],0}^r(x)$. If $G_0(x)$ is partially complex, then $\mathfrak{N}_x \subset (\mathcal{O}_x + \tau\mathcal{O}_x)$ by Theorem 8.9, so $S_{[x]}$ is open in Y by Lemma 8.4. If $S_{[x]}$ is open in Y , then it is a complex submanifold of X , so $G_0(x)$ is partially complex.

Let $G_0(x)$ be partially complex and of flag type. Then $\mathfrak{N}_{[x]} = (\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x + \mathfrak{w}_x$ where \mathfrak{w}_x is a τ -stable sum of root spaces \mathfrak{G}_φ , $\varphi \in (\Phi^r \cap -\tau\Phi^u) \cup (-\Phi^u \cap \tau\Phi^r)$. Now $\dim_C S_{[x]} = \dim_C N_{[x]}(x) = \dim_C \mathfrak{U}_x + \dim_C (\mathfrak{W}_x \cap \tau\mathcal{O}_x) = |\Phi^u \cap -\tau\Phi^u| + \dim_C (\mathfrak{W}_x \cap \tau\mathcal{O}_x) \cong |\Phi^u \cap -\tau\Phi^u|$, and equality means $\mathfrak{w}_x = 0$, i.e. $\mathfrak{N}_{[x]} = (\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x$, i.e. $G_0(x)$ measurable. Q.E.D.

9.8. COROLLARY. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , $x \in X$, $\mathfrak{K} \subset \mathcal{O}_x$ a τ -stable Cartan subalgebra of \mathfrak{g} , Φ a subset of a simple \mathfrak{K} -root system such that $\mathcal{O}_x = \mathcal{O}_\Phi$, and \mathfrak{U}_x as in (9.3).*

1. $G_0(x)$ is of flag type relative to its algebraic arc components if and only if $\mathfrak{U}_x \subset \mathcal{Q}_{[x]}$.

2. If $G_0(x)$ is of flag type relative to its algebraic arc components, then (2a) $G_0(x)$ is of flag type,

(2b) the algebraic arc component of $G_0(x)$ through x is an open $\mathcal{Q}_{[x],0}$ -orbit on the complex flag manifold $\mathcal{Q}_{[x]}(x)$,

(2c) the algebraic arc components of $G_0(x)$ have complex dimension $\cong |\Phi^u \cap -\tau\Phi^u|$, with equality if and only if $G_0(x)$ is measurable relative to its algebraic arc components.

PROOF. $\mathcal{Q}_{[x]}$ can be substituted for $\mathfrak{N}_{[x]}$ throughout the proof of Theorem 9.7. Q.E.D.

Recall Example 8.12. There $G = \text{SL}(2m, \mathbf{C})$, G_0 is $\text{SL}(m, \mathbf{Q})$ or $\text{SL}(2m, \mathbf{R})$, the simple root system $\Psi = \{\psi_1, \dots, \psi_{2m-1}\}$, $\tau\psi_i = \psi_{2m-i}$, and $\mathcal{O}_x = \mathcal{O}_\Phi$ where $\Phi = \{\psi_1, \dots, \psi_{2m-2}\}$. So X is complex projective $(2m-1)$ -space and we calculated

$$\mathcal{Q}_{[x]} = \mathcal{O}_\Gamma \text{ where } \Gamma = \{\psi_2, \dots, \psi_{2m-2}\} \text{ and}$$

$$\mathfrak{N}_{[x]} = \mathcal{O}_x + \tau\mathcal{O}_x = \sum_{\varphi \neq -\mu} \mathfrak{G}_\varphi \text{ where } \mu = \psi_1 + \psi_2 + \dots + \psi_{2m-1}.$$

Thus $\mathfrak{N}_{[x]}$ generates \mathfrak{g} . As $\mathfrak{M}_{[x]} \subset \mathfrak{N}_{[x]}$ by Proposition 8.7, it follows that $\mathfrak{N}_{[x]} = \mathfrak{g}$. Thus $G_0(x)$ is of flag type, but it is not partially complex.

9.9. THEOREM. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , and $x \in X$ such that the orbit $G_0(x)$ is polarized. Then the following conditions are equivalent:*

- (i) $G_0(x)$ is integrable.
- (ii) $G_0(x)$ is measurable.
- (iii) $G_0(x)$ is measurable relative to its algebraic arc components.
- (iv) $G_0(x)$ is partially complex and of flag type.
- (v) $G_0(x)$ is partially complex and is of flag type relative to its algebraic arc components.

If the above conditions hold, then $\mathfrak{N}_{[x]} = (\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x = \mathcal{Q}_{[x]}$, so the algebraic arc components of $G_0(x)$ coincide with the holomorphic arc components.

PROOF. If $G_0(x)$ is measurable, Theorem 9.2 says that it is integrable. If $G_0(x)$ is measurable relative to its algebraic arc components, Corollary 9.5 says that it is integrable.

Suppose $G_0(x)$ integrable, so (8.6) and Theorem 8.9 show that $\mathfrak{N}_{[x]} = (\mathcal{O}_x + \tau\mathcal{O}_x) = \mathcal{Q}_{[x]}$. As $G_0(x)$ is polarized, $(\mathcal{O}_x + \tau\mathcal{O}_x) = (\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x$. Thus $\mathfrak{N}_{[x]} = (\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x = \mathcal{Q}_{[x]}$, and Theorem 9.2 says that $G_0(x)$ both is measurable, and is measurable relative to its algebraic arc components.

If $G_0(x)$ is measurable, Theorem 9.2 says it is partially complex and of flag type. If $G_0(x)$ is partially complex and of flag type, Theorems 8.9 and 9.7 say that $(\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x \subset \mathfrak{N}_{[x]} \subset (\mathcal{O}_x \cap \tau\mathcal{O}_x)$, while $(\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x = (\mathcal{O}_x + \tau\mathcal{O}_x)$ because $G_0(x)$ is polarized; thus $\mathfrak{N}_{[x]} = (\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x$ and $G_0(x)$ is measurable by Theorem 9.2.

If $G_0(x)$ is measurable relative to its algebraic arc components, then it is measurable because it is polarized, so it is partially complex by Theorem 9.2; and $\mathfrak{U}_x \subset \mathcal{O}_{[x]}$ by Corollary 9.5, so $G_0(x)$ is of flag type relative to its algebraic arc components by Corollary 9.8. If $G_0(x)$ is partially complex and is of flag type relative to its algebraic arc components, Theorem 8.9 and Corollary 9.8 say that $(\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x \subset \mathcal{Q}_{[x]} \subset \mathfrak{N}_{[x]} \subset (\mathcal{O}_x + \tau\mathcal{O}_x)$, while $(\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x = (\mathcal{O}_x + \tau\mathcal{O}_x)$ because $G_0(x)$ is polarized; thus $\mathcal{Q}_{[x]} = (\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x$ and Corollary 9.5 says that $G_0(x)$ is measurable relative to its algebraic arc components.
Q.E.D.

9.10. COROLLARY. Let B be a Borel subgroup of G , $Y = G/B$ a complex flag manifold, G_0 a real form of G , and $y \in Y$. Then the following conditions are equivalent, and they imply that the algebraic arc components of $G_0(y)$ coincide with the holomorphic arc components:

- (i) $G_0(y)$ is integrable.
- (ii) $G_0(y)$ is measurable.
- (iii) $G_0(y)$ is measurable relative to its algebraic arc components.
- (iv) $G_0(y)$ is partially complex and of flag type.

(v) $G_0(y)$ is partially complex and is of flag type relative to its algebraic arc components.

PROOF. Let $\mathfrak{K} \subset \mathfrak{B}_y$ be a τ -stable Cartan subalgebra of \mathfrak{g} . Then $\mathfrak{B}'_y = \mathfrak{K} = \tau\mathfrak{K} = \tau\mathfrak{B}'_y$, so $G_0(y)$ is polarized and Theorem 9.9 applies. Q.E.D.

9.11. COROLLARY. Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , and $x \in X$ such that the orbit $G_0(x)$ is integrable.

1. $\mathfrak{H}_{[x]} = (\mathfrak{O}_x + \tau\mathfrak{O}_x) = \mathfrak{Q}_{[x]}$, so the algebraic arc components of $G_0(x)$ coincide with the holomorphic arc components.

2. $G_0(x)$ is partially complex and of flag type.

3. $G_0(x)$ is measurable if and only if it is polarized.

PROOF. Assertion (1) is immediate from (8.6) and Theorem 8.9. Now Theorem 8.9 proves $G_0(x)$ partially complex and Theorem 9.7 says that it is of flag type. If $G_0(x)$ is measurable, Theorem 9.2 says it is polarized; if $G_0(x)$ is polarized, Theorem 9.9 says it is measurable.

Q.E.D.

If $G_0(x)$ is an open orbit, then $S_{[x]} = G_0(x)$, so it is rather trivial that $G_0(x)$ is partially complex and of flag type. Criteria for measurability of $G_0(x)$ are given by Theorems 6.3 and 6.7, the latter being that \mathfrak{O} and $\tau\mathfrak{O}$ are opposed. We now see that closed orbits are somewhat better behaved.

9.12. THEOREM. Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , and $x \in X$ such that $G_0(x)$ is the closed G_0 -orbit on X . Then $G_0(x)$ is measurable, hence partially complex and of flag type.

PROOF. Let $B \subset P$ be a Borel subgroup of G , $Y = G/B$ the resulting complex flag manifold, and $\pi: Y \rightarrow X$ the projection. Choose $y \in \pi^{-1}(x)$ such that $G_0(y)$ is the closed G_0 -orbit on Y .

Let \mathfrak{K}_0 be the Lie algebra of a maximally split Cartan subgroup of G_0 , $\mathfrak{K}_0 = \mathfrak{J} + \mathfrak{A}$ the decomposition into toroidal and vector parts, and $\mathfrak{K} = \mathfrak{K}_0^{\mathbb{C}}$. The \mathfrak{K} -root system $\Delta = \Delta_0 \cup \Delta_r$, where Δ_0 consists of those roots that vanish on \mathfrak{A} and Δ_r consists of those that do not. Ordering the roots, we have

$$\text{Iwasawa decomposition } \mathfrak{G}_0 = \mathfrak{K} + \mathfrak{A} + \mathfrak{N}, \quad \mathfrak{N}^{\mathbb{C}} = \sum_{\Delta_r^+} \mathfrak{G}_{\rho}$$

$$\text{minimal parabolic } \mathfrak{L} = \mathfrak{M} + \mathfrak{A} + \mathfrak{N}, \quad \mathfrak{M}^{\mathbb{C}} = \mathfrak{J}^{\mathbb{C}} + \sum_{\Delta_0} \mathfrak{G}_{\rho}$$

Now define a Borel subalgebra $\mathfrak{B}' \subset \mathfrak{g}$ by

$$\mathfrak{B}' = (\mathfrak{J} + \mathfrak{A} + \mathfrak{N})^{\mathbb{C}} + \sum_{\Delta_r^+} \mathfrak{G}_{\rho}$$

Then

$$\tau\mathfrak{B}' = (\mathfrak{J} + \mathfrak{A} + \mathfrak{N})^G + \sum_{\Delta_0^-} \mathfrak{G}_\varphi.$$

Thus

(i) $\mathfrak{B}' \cap \tau\mathfrak{B}' = (\mathfrak{J} + \mathfrak{A} + \mathfrak{N})^G$ and

(ii) $\mathfrak{B}' + \tau\mathfrak{B}' = \mathfrak{L}^G.$

Let $y' \in Y$ be the point such that $\mathfrak{B}_{y'} = \mathfrak{B}'$. Then (ii) says that $G_0(y')$ is integrable, hence measurable by Corollary 9.10, hence partially complex and of flag type by Theorem 9.2 or Corollary 9.10. On the other hand, (i) shows that the compact group K with Lie algebra \mathfrak{K} is transitive on $G_0(y')$. Thus $G_0(y')$ is the closed G_0 -orbit on Y , i.e. $G_0(y') = G_0(y)$. That proves our assertions for the case where P is a Borel subgroup of G .

Conjugate \mathfrak{B}' by an element of G_0 that sends y' to y . That done, we may assume that \mathfrak{B}_y is given by the description of \mathfrak{B}' above. Now $\mathcal{P}_x = \mathfrak{B}_y + \mathfrak{D}_0 + \mathfrak{D}_r$ where, for certain subsets $D_0 \subset \Delta_0^-$ and $D_r \subset \Delta_r^-$,

$$\mathfrak{D}_0 = \sum_{D_0} \mathfrak{G}_\varphi \quad \text{and} \quad \mathfrak{D}_r = \sum_{D_r} \mathfrak{G}_\varphi.$$

Thus

$$\mathcal{P}_x = \mathfrak{B}_y + \mathfrak{D}_0 + \mathfrak{D}_r \quad \text{and} \quad \mathfrak{B}_y^{-u} = \mathcal{P}_x^{-u} + \mathfrak{D}_0 + \mathfrak{D}_r,$$

and, defining

$$\mathfrak{D}_0^- = \sum_{-D_0} \mathfrak{G}_\varphi, \quad \mathfrak{D}_r^- = \sum_{-D_r} \mathfrak{G}_\varphi, \quad \mathfrak{N}^- = \left\{ \sum_{\Delta_r^-} \mathfrak{G}_\varphi \right\} \cap \mathfrak{G}_0,$$

we have

$$\tau\mathfrak{D}_0 = \mathfrak{D}_0^- \subset \mathfrak{B}_y \quad \text{and} \quad \mathfrak{D}_r + \tau\mathfrak{D}_r \subset (\mathfrak{N}^-)^G.$$

In particular,

$$\mathfrak{D}_r \cap \tau\mathfrak{B}_y^u = \tau\mathfrak{D}_r \cap \mathfrak{B}_y^u = \mathfrak{D}_r^- \cap \tau\mathfrak{B}_y^{-u} = \tau\mathfrak{D}_r^- \cap \mathfrak{B}_y^{-u} = 0.$$

Retaining the notation (9.3), we now have

$$\begin{aligned} \mathfrak{V}_y &= (\mathfrak{B}_y^u \cap \tau\mathfrak{B}_y^{-u}) + (\mathfrak{B}_y^{-u} \cap \tau\mathfrak{B}_y^u) \\ &= \{(\mathcal{P}_x^u + \mathfrak{D}_0^-) \cap (\tau\mathcal{P}_x^{-u} + \tau\mathfrak{D}_0)\} + \{(\mathcal{P}_x^{-u} + \mathfrak{D}_0) \cap (\tau\mathcal{P}_x^u + \tau\mathfrak{D}_0^-)\} \\ &= (\mathcal{P}_x^u \cap \tau\mathcal{P}_x^{-u}) + (\mathcal{P}_x^{-u} \cap \tau\mathcal{P}_x^u) + \mathfrak{D}_0 + \mathfrak{D}_0^- = \mathfrak{V}_x + \mathfrak{D}_0 + \mathfrak{D}_0^-. \end{aligned}$$

We also compute

$$\begin{aligned} (\mathcal{P}_x \cap \tau\mathcal{P}_x) &= (\mathfrak{B}_y + \mathfrak{D}_0 + \mathfrak{D}_r) \cap (\tau\mathfrak{B}_y + \tau\mathfrak{D}_0 + \tau\mathfrak{D}_r) \\ &= (\mathfrak{B}_y \cap \tau\mathfrak{B}_y) + (\mathfrak{D}_0 + \mathfrak{D}_0^-) + (\mathfrak{D}_r \cap \tau\mathfrak{D}_r). \end{aligned}$$

As $G_0(y)$ is measurable, $\mathfrak{N}_{[y]} = (\mathfrak{B}_y \cap \tau\mathfrak{B}_y) + \mathfrak{U}_y$ by Theorem 9.2. Now define

$$(9.13a) \quad \mathfrak{W}_{[x]} = (\mathcal{P}_x \cap \tau\mathcal{P}_x) + \mathfrak{U}_x.$$

Our calculations above say that

$$\begin{aligned} \mathfrak{W}_{[x]} &= (\mathfrak{B}_y \cap \tau\mathfrak{B}_y) + (\mathfrak{D}_0 + \mathfrak{D}_0^-) + (\mathfrak{D}_r \cap \tau\mathfrak{D}_r) + \mathfrak{U}_x \\ &= (\mathfrak{B}_y \cap \tau\mathfrak{B}_y) + \mathfrak{U}_y + (\mathfrak{D}_r \cap \tau\mathfrak{D}_r). \end{aligned}$$

In other words,

$$(9.13b) \quad \mathfrak{W}_{[x]} = \mathfrak{N}_{[y]} + (\mathfrak{D}_r \cap \tau\mathfrak{D}_r).$$

As both $\mathfrak{N}_{[y]}$ and $(\mathcal{P}_x \cap \tau\mathcal{P}_x)$ are subalgebras of $\mathfrak{W}_{[x]}$, and $(\mathcal{P}_x \cap \tau\mathcal{P}_x) + \mathfrak{U}_x = \mathfrak{W}_{[x]} = \mathfrak{N}_{[y]} + (\mathfrak{D}_r \cap \tau\mathfrak{D}_r)$ by (9.13 a, b), the condition for $\mathfrak{W}_{[x]}$ to be an algebra is

$$[\mathfrak{U}_x, \mathfrak{D}_r \cap \tau\mathfrak{D}_r] \subset \mathfrak{W}_{[x]}.$$

Observing $\mathfrak{D}_r \subset \mathcal{P}_x^r$, we calculate

$$\begin{aligned} [\mathfrak{U}_x^+, \mathfrak{D}_r \cap \tau\mathfrak{D}_r] &\subset [\mathcal{P}_x^u \cap \tau\mathcal{P}_x^{-u}, \mathcal{P}_x^r \cap \tau\mathcal{P}_x^r] \\ &\subset \{[\mathcal{P}_x^u, \mathcal{P}_x^r] \cap [\tau\mathcal{P}_x^{-u}, \tau\mathcal{P}_x^r]\} \subset (\mathcal{P}_x^u \cap \tau\mathcal{P}_x^{-u}) \subset \mathfrak{U}_x^+, \end{aligned}$$

and similarly $[\mathfrak{U}_x^-, \mathfrak{D}_r \cap \tau\mathfrak{D}_r] \subset \mathfrak{U}_x^-$. Thus $[\mathfrak{U}_x, \mathfrak{D}_r \cap \tau\mathfrak{D}_r] \subset \mathfrak{U}_x \subset \mathfrak{W}_{[x]}$. That proves $\mathfrak{W}_{[x]}$ to be an algebra. As it contains the parabolic subalgebra $\mathfrak{N}_{[y]}$ of \mathfrak{g} , and as $\tau\mathfrak{W}_{[x]} = \mathfrak{W}_{[x]}$, now

$$(9.13c) \quad \mathfrak{W}_{[x]} \text{ is a } \tau\text{-stable parabolic subalgebra of } \mathfrak{g}.$$

Comparing (9.4) and (9.13a), Theorem 9.2 says that $G_0(x)$ is measurable if and only if $\mathfrak{W}_{[x]} = \mathfrak{N}_{[x]}$. As $\mathfrak{W}_{[x]} \subset \mathfrak{N}_{[x]}$ by Lemma 8.4 and Proposition 8.7, Theorem 8.9 says we need only check that $\mathfrak{W}_{[x]} = \mathfrak{N}_{[x]}$. Define A to be the set of roots such that $\mathfrak{N}_{[x]} = \mathfrak{W}_{[x]} + \sum_A \mathfrak{G}_\varphi$ (direct); now the assertion of our theorem is reduced to the statement that A is empty.

Express $\mathfrak{W}_{[x]}^u = \sum_M \mathfrak{G}_\varphi$ and define $\mu = \sum_M \varphi$. Then (9.13c) says that a root space $\mathfrak{G}_\varphi \subset \mathfrak{W}_{[x]}^u$ if and only if $\langle \varphi, \mu \rangle = 0$, $\mathfrak{G}_\varphi \subset \mathfrak{W}_{[x]}^u$ if and only if $\langle \varphi, \mu \rangle > 0$, and $\mu = \tau\mu$. In particular, $\langle \varphi, \mu \rangle < 0$ for every $\varphi \in A$, and $\langle \varphi, \mu \rangle \geq 0$ whenever $\mathfrak{G}_\varphi \subset (\mathcal{P}_x \cap \tau\mathcal{P}_x)$.

Let $\varphi \in A$. If $\varphi + \tau\varphi$ is a root, then $\mathfrak{G}_{\varphi + \tau\varphi} \subset (\mathcal{P}_x + \tau\mathcal{P}_x)$ implies $\mathfrak{G}_{\varphi + \tau\varphi} \subset (\mathcal{P}_x \cap \tau\mathcal{P}_x)$ because $\tau(\varphi + \tau\varphi) = \varphi + \tau\varphi$; then $0 \leq \langle \varphi + \tau\varphi, \mu \rangle = \langle \varphi, \mu \rangle + \langle \tau\varphi, \mu \rangle = \langle \varphi, \mu \rangle + \langle \tau\varphi, \tau\mu \rangle = 2\langle \varphi, \mu \rangle < 0$, which is absurd; thus $\mathfrak{G}_{\varphi + \tau\varphi} \subset \mathcal{P}_x^{-u} \cap \tau\mathcal{P}_x^{-u}$, so φ is not an \mathfrak{K} -root of $\mathfrak{N}_{[x]}$; that contradicts $\varphi \in A$.

Let $\varphi \in A$. Now $\varphi + \tau\varphi$ is not a root. As $(\mathfrak{B}_y + \tau\mathfrak{B}_y) = \mathfrak{N}_{[y]} \subset \mathfrak{W}_{[x]}$ because $G_0(y)$ is integrable, we have $\mathfrak{G}_\varphi \not\subset (\mathfrak{B}_y + \tau\mathfrak{B}_y)$. However $\mathfrak{G}_\varphi \subset (\mathcal{P}_x + \tau\mathcal{P}_x)$. Replace φ by $\tau\varphi$ if necessary so $\mathfrak{G}_\varphi \subset \mathcal{P}_x = \mathfrak{B}_y + \mathfrak{D}_0 + \mathfrak{D}_r$. As $\mathfrak{D}_0 \subset \tau\mathfrak{B}_y$ now $\mathfrak{G}_\varphi \subset \mathfrak{D}_r$. As $\mathfrak{D}_r \cap \tau\mathfrak{D}_r \subset \mathfrak{W}_{[x]}$ now $\mathfrak{G}_\varphi \subset \tau\mathcal{P}_x$. Thus

$\mathfrak{G}_\varphi \subset \mathcal{O}'_x \cap \tau \mathcal{O}_x^{-u}$. Also, $\mathfrak{G}_\varphi \subset \mathcal{D}_\tau$ shows $\langle \varphi, \tau\varphi \rangle \neq 0$, so Proposition 8.7(2) gives us the fact that

$$(9.14a) \quad \mathbf{Z} = h_\varphi \mathbf{C} + h_{\tau\varphi} \mathbf{C} + \mathfrak{G}_\varphi + \mathfrak{G}_{\tau\varphi} + \mathfrak{G}_{-\varphi} + \mathfrak{G}_{-\tau\varphi} + \mathfrak{G}_{\tau\varphi-\varphi} + \mathfrak{G}_{\varphi-\tau\varphi}$$

is a τ -stable subalgebra of type A_2 in \mathfrak{G} that is contained in $\mathfrak{M}_{[x]}$ and has Cartan subalgebra $\mathfrak{K}' = h_\varphi \mathbf{C} + h_{\tau\varphi} \mathbf{C}$. Define real forms $\mathbf{Z}_0 = \mathbf{Z} \cap \mathfrak{G}_0$ and $\mathfrak{K}'_0 = \mathfrak{K}' \cap \mathfrak{G}_0 = \mathfrak{K} \cap \mathbf{Z}_0$. Then $\mathfrak{K}'_0 = \mathfrak{J}' + \mathfrak{A}'$ where

$$(9.14b) \quad \mathfrak{J}' = \mathfrak{J} \cap \mathfrak{K}' = i(h_\varphi - h_{\tau\varphi})\mathbf{R} \text{ and } \mathfrak{A}' = \mathfrak{A} \cap \mathfrak{K}' = (h_\varphi + h_{\tau\varphi})\mathbf{R}.$$

Note that \mathfrak{J}' contains a regular element $i(h_\varphi - h_{\tau\varphi})$ of \mathbf{Z} . Now Lemma 4.1 says that \mathfrak{K}'_0 is a maximally compact Cartan subalgebra of \mathbf{Z}_0 . As \mathbf{Z} is of type A_2 , i.e. $\mathbf{Z} = \mathfrak{S}\mathfrak{L}(3, \mathbf{C})$, its real forms are $\mathfrak{S}\mathfrak{L}(3, \mathbf{R})$ and the Lie algebras of the groups $\mathbf{SU}(3)$ and $\mathbf{SU}^1(3)$. In the latter two cases, a maximally compact Cartan subalgebra of the real form is the Lie algebra of a (compact) torus. Thus $\mathbf{Z}_0 \subset \mathbf{Z}$ is $\mathfrak{S}\mathfrak{L}(3, \mathbf{R}) \subset \mathfrak{S}\mathfrak{L}(3, \mathbf{C})$. That implies that \mathbf{Z}_0 has a split Cartan subalgebra \mathfrak{A}'' . Now consider the algebra $\mathfrak{K}'_0 + \mathbf{Z}_0$. It is reductive, and \mathfrak{K}'_0 is its maximally split Cartan subalgebra. Now there exists an element g in the analytic subgroup of G_0 for $\mathfrak{K}'_0 + \mathbf{Z}_0$, such that $\text{ad}(g)\mathfrak{A}'' \subset \mathfrak{A}$. Replacing x by $g^{-1}(x)$ we obtain (9.14a) with $\mathfrak{K}'_0 = \mathfrak{A}''$ split, contradicting (9.14b). That contradicts $\varphi \in A$.

We have proved A empty, so $\mathfrak{M}_{[x]} = \mathfrak{W}_{[x]}$ algebra, whence $\mathfrak{N}_{[x]} = \mathfrak{W}_{[x]}$ by Theorem 8.9, so $G_0(x)$ is measurable by (9.13a) and Theorem 9.2. Q.E.D.

9.15. COROLLARY. *Let $P_1 \subset P_2$ be parabolic subgroups of G , $X_i = G/P_i$ the resulting complex flag manifolds, and $\pi: X_1 \rightarrow X_2$ the projection $gP_1 \rightarrow gP_2$. Let G_0 be a real form of G . Choose $x_i \in X_i$ such that $\pi(x_1) = x_2$ and $G_0(x_i)$ is the closed G_0 -orbit on X_i . Then holomorphic arc components satisfy $\pi S_{[x_1]} = S_{[x_2]}$.*

PROOF. Let $B \subset P_1$ be a Borel subgroup of G , $Y = G/B$, and $\nu_i: Y \rightarrow X_i$ the projections. Then $\nu_2 = \pi \cdot \nu_1$. Let $y \in \nu_1^{-1}(x_1)$ such that $G_0(y)$ is the closed G_0 -orbit on Y . Then $\nu_i(y) = x_i$. Recall (9.13b) from the proof of Theorem 9.12; it says $\mathfrak{W}_{[x_i]} = \mathfrak{N}_{[y]} + (\text{a certain subspace of } (\mathfrak{O}_i)_{x_i} \cap \tau(\mathfrak{O}_i)_{x_i})$. The subsequent portion of the proof of Theorem 9.12 consisted of showing $\mathfrak{W}_{[x_i]} = \mathfrak{N}_{[x_i]}$. Now $\mathfrak{N}_{[x_i]} = \mathfrak{N}_{[y]} + (\text{subspace of } (\mathfrak{O}_i)_{x_i} \cap \tau(\mathfrak{O}_i)_{x_i})$. That proves $\nu_i S_{[y]} = S_{[x_i]}$. As $\pi = \nu_2 \cdot \nu_1^{-1}$, it follows that $\pi S_{[x_1]} = S_{[x_2]}$. Q.E.D.

The (rather minor) portion of Theorem 9.12 which says that closed orbits are partially complex, allows us to strengthen Corollary 8.19.

9.16. COROLLARY. *Let $X = G/P$ be a complex flag manifold, $x \in X$, and G_0 a real form of G . Then the following conditions are equivalent.*

- (i) $G_0(x)$ is the closed G_0 -orbit on X .
- (ii) The holomorphic arc components of $G_0(x)$ are compact.
- (iii) The holomorphic arc components of $G_0(x)$ are complex flag manifolds.
- (iv) $\mathfrak{N}_{[x]}^u \subset (\mathcal{O}_x \cap \tau\mathcal{O}_x)$, and $\mathfrak{N}_{[x]}^r = \mathfrak{L} \oplus \mathfrak{g}$ direct sum of τ -stable ideals, where $\mathfrak{g} \subset (\mathcal{O}_x \cap \tau\mathcal{O}_x)$ and $\mathfrak{L} \cap \mathfrak{G}_0$ is a compactly embedded subalgebra of \mathfrak{G}_0 .

PROOF. Corollary 8.19 gives equivalence of (i) and (ii). As complex flag manifolds are compact, (iii) implies (ii). Conversely (ii) implies (iii) because $G_0(x)$ is of flag type and $S_{[x]}$ is an open $N_{[x],0}^r$ -orbit on the complex flag $N_{[x]}^r(x)$.

Given (iv), $\mathfrak{g} + \mathfrak{N}_{[x]}^u$ is an ideal of $\mathfrak{N}_{[x]}$ contained in the isotropy subalgebra at x , so the subgroup $J_0 \cdot N_{[x],0}^u$ of G_0 for the subalgebra $(\mathfrak{g} + \mathfrak{N}_{[x]}^u) \cap \mathfrak{G}_0$ must act trivially on $S_{[x]}$. Thus the subgroup L_0 for $\mathfrak{L} \cap \mathfrak{G}_0$ is transitive on $S_{[x]}$. But $\mathfrak{L} \cap \mathfrak{G}_0$ is compactly embedded, and G_0 has finite center, so L_0 is compact. Thus $S_{[x]}$ is compact. Now (iv) implies (ii).

Given (iii), we have $\mathfrak{N}_{[x]}^u \subset (\mathcal{O}_x \cap \tau\mathcal{O}_x)$ and $N_{[x],0}^r$ transitive on $S_{[x]}$. Let J_0 be the kernel of its action, so the Lie algebra \mathfrak{g}_0 is an ideal of $\mathfrak{N}_{[x],0}^r$ contained in $(\mathcal{O}_x \cap \tau\mathcal{O}_x)_0$. Let \mathfrak{L}_0 be a τ -stable complementary ideal, L_0 the corresponding subgroup of $N_{[x],0}^r$. Then L_0 is transitive on the flag manifold $S_{[x]}$, acting with finite kernel, so L_0 is compact. Now (iv) follows with $\mathfrak{L} = \mathfrak{L}_0^C$ and $\mathfrak{g} = \mathfrak{g}_0^C$. Q.E.D.

We can now complete the results of Corollary 8.20.

9.17. COROLLARY. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , and $x \in X$. Then the holomorphic arc components of $G_0(x)$ are points if and only if (a) $G_0(x)$ is the closed orbit and (b) $\tau\mathcal{O}_x^u \subset \mathcal{O}_x$.*

PROOF. If the holomorphic arc components of $G_0(x)$ are points, they are compact, so $G_0(x)$ is the closed orbit by Corollary 8.19 or Corollary 9.16. Now let $G_0(x)$ be the closed orbit, so $\mathfrak{N}_{[x]} = (\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x$ by Theorem 9.2; thus $S_{[x]}$ is a point if and only if $\mathfrak{U}_x = 0$. As $\mathfrak{U}_x = (\mathcal{O}_x^u \cap \tau\mathcal{O}_x^{-u}) + \tau(\mathcal{O}_x^u \cap \tau\mathcal{O}_x^{-u})$, it is 0 if and only if $\tau\mathcal{O}_x^u \cap \mathcal{O}_x^{-u} = 0$. Thus $S_{[x]}$ is a point if and only if $\tau\mathcal{O}_x^u \subset \mathcal{O}_x$. Q.E.D.

It turns out that the proof of Theorem 9.12 does not apply exclusively to the case where $G_0(x)$ is closed.

9.18. PROPOSITION. *Let $X = G/P$ be a complex flag manifold, G_0 a real form of G , and $x \in X$. Suppose that $\mathfrak{G}_0 \cap \mathcal{O}_x$ contains a maximally split Cartan subalgebra \mathfrak{H}_0 of \mathfrak{G}_0 . Then the following conditions are equivalent:*

- (i) $G_0(x)$ is measurable.
- (ii) $\mathfrak{N}_{[x]} = (\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x$.
- (iii) $(\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathfrak{U}_x$ is an algebra.

PROOF. Theorem 9.2 is the equivalence of (i) and (ii), and of course (ii) implies (iii). Thus we need only prove that (iii) implies (ii).

Assume (iii) and define $\mathfrak{W}_{[x]} = (\mathcal{O}_x \cap \tau\mathcal{O}_x) + \mathcal{U}_x$. The latter agrees with (9.13a). Now $\mathfrak{W}_{[x]}$ is an algebra. As $\tau\mathfrak{W}_{[x]} = \mathfrak{W}_{[x]}$ and $(\mathcal{O}_x \cap \tau\mathcal{O}_x) \subset \mathfrak{W}_{[x]} \subset (\mathcal{O}_x + \tau\mathcal{O}_x)$, Lemma 8.4 and Proposition 8.7 say $\mathfrak{W}_{[x]} \subset \mathfrak{M}_{[x]}$. Here we work with the Cartan subalgebra $\mathfrak{H} = \mathfrak{H}_0^{\mathcal{C}}$ of \mathfrak{G} . Define A to be the set of \mathfrak{H} -roots such that $\mathfrak{M}_{[x]}$ is the direct sum of $\mathfrak{W}_{[x]}$ and $\sum_A \mathfrak{G}_{\varphi}$. As in Theorem 9.12, we will prove that A is empty.

Let φ be a root with $\mathfrak{G}_{\varphi} \not\subset \mathfrak{W}_{[x]}$. Then φ must be in $\Phi^r \cap -\tau\Phi^u$, $-\Phi^u \cap \tau\Phi^r$ or $-\Phi^u \cap -\tau\Phi^u$ where $\mathcal{O}_x = \mathcal{O}_{\Phi}$. Thus $\mathfrak{G}_{-\varphi} \subset \mathfrak{W}_{[x]}$. That proves $\mathfrak{W}_{[x]}$ parabolic in \mathfrak{G} . Define μ to be the sum of all roots φ with $\mathfrak{G}_{\varphi} \subset \mathfrak{W}_{[x]}^u$. Then $\mu = \tau\mu$ and, for any root φ ,

$$\begin{aligned} \text{if } \langle \varphi, \mu \rangle > 0 & \quad \text{then } \mathfrak{G}_{\varphi} \subset \mathfrak{W}_{[x]}^u, \\ \text{if } \langle \varphi, \mu \rangle = 0 & \quad \text{then } \mathfrak{G}_{\varphi} \subset \mathfrak{W}_{[x]}^r, \\ \text{if } \langle \varphi, \mu \rangle < 0 & \quad \text{then } \mathfrak{G}_{\varphi} \not\subset \mathfrak{W}_{[x]}. \end{aligned}$$

In particular, $\langle \varphi, \mu \rangle < 0$ for every $\varphi \in A$, and $\langle \varphi, \mu \rangle \geq 0$ whenever $\mathfrak{G}_{\varphi} \subset (\mathcal{O}_x \cap \tau\mathcal{O}_x)$.

Let $\varphi \in A$. If $\varphi + \tau\varphi$ is a root, then $\mathfrak{G}_{\varphi + \tau\varphi} \subset (\mathcal{O}_x + \tau\mathcal{O}_x)$ would imply $\mathfrak{G}_{\varphi + \tau\varphi} \subset (\mathcal{O}_x \cap \tau\mathcal{O}_x)$, so $0 \leq \langle \varphi + \tau\varphi, \mu \rangle = 2\langle \varphi, \mu \rangle < 0$, which is absurd; and $\mathfrak{G}_{\varphi + \tau\varphi} \not\subset (\mathcal{O}_x + \tau\mathcal{O}_x)$ would contradict $\mathfrak{G}_{\varphi} \subset \mathfrak{M}_{[x]}$; thus $\varphi + \tau\varphi$ is not a root. As in Proposition 8.7, it follows that there are just two possibilities, given as follows.

Case 1. $\varphi \neq \tau\varphi \neq -\varphi$, $\langle \varphi, \tau\varphi \rangle = 0$, $\tau\varphi - \varphi$ is not a root, and

$$(9.19a) \quad \mathbf{Z} = h_{\varphi}\mathbf{C} + h_{\tau\varphi}\mathbf{C} + \mathfrak{G}_{\varphi} + \mathfrak{G}_{-\varphi} + \mathfrak{G}_{\tau\varphi} + \mathfrak{G}_{-\tau\varphi}$$

is a τ -stable semisimple subalgebra of type $A_1 \oplus A_1$ in $\mathfrak{W}_{[x]}$.

Case 2. $\varphi \neq \tau\varphi \neq -\varphi$, $\langle \varphi, \tau\varphi \rangle = \frac{1}{2}\langle \varphi, \varphi \rangle$, $\tau\varphi - \varphi$ is a root, and

$$(9.19b) \quad \mathbf{Z} = h_{\varphi}\mathbf{C} + h_{\tau\varphi}\mathbf{C} + \mathfrak{G}_{\varphi} + \mathfrak{G}_{-\varphi} + \mathfrak{G}_{\tau\varphi} + \mathfrak{G}_{-\tau\varphi} + \mathfrak{G}_{\tau\varphi - \varphi} + \mathfrak{G}_{-\tau\varphi - \varphi}$$

is a τ -stable semisimple subalgebra of type A_2 in $\mathfrak{W}_{[x]}$.

In both cases, $\mathfrak{H}' = h_{\varphi}\mathbf{C} + h_{\tau\varphi}\mathbf{C}$ is a Cartan subalgebra of \mathbf{Z} , and we define real forms $\mathbf{Z}_0 = \mathbf{Z} \cap \mathfrak{G}_0$ and $\mathfrak{H}'_0 = \mathfrak{H}' \cap \mathbf{Z}_0 = \mathfrak{H}' \cap \mathfrak{G}_0$. Then, as in (9.14b), $\mathfrak{H}'_0 = \mathfrak{J}' + \mathfrak{A}'$ where

$$(9.19c) \quad \mathfrak{J}' = i(h_{\varphi} - h_{\tau\varphi})\mathbf{R} \quad \text{and} \quad \mathfrak{A}' = (h_{\varphi} + h_{\tau\varphi})\mathbf{R}.$$

Checking the two cases separately one sees that $i(h_{\varphi} - h_{\tau\varphi}) \in \mathfrak{J}'$ is a regular element of \mathbf{Z} , so Lemma 4.1 assures us that \mathfrak{H}'_0 is a maximally compact Cartan subalgebra of \mathbf{Z}_0 . We now use the hypothesis that \mathfrak{H}_0 is maximally split in \mathfrak{G}_0 ; it allows us to use the argument following (9.14b), and this eliminates Case 2.

Now we are in Case 1 with Z given by (9.19a). As $\varphi \in A$ we have $\mathfrak{g}_\varphi \subset (\mathcal{P}_x + \tau\mathcal{P}_x)$. Replace φ by $\tau\varphi$ if necessary for the normalization $\mathfrak{g}_\varphi \subset \mathcal{P}_x$. As $\langle \varphi, \mu \rangle < 0$ implies $\mathfrak{g}_\varphi \not\subset (\mathcal{P}_x \cap \tau\mathcal{P}_x)$ we have $\mathfrak{g}_\varphi \subset (\mathcal{P}_x \cap \tau\mathcal{P}_x^-)$. Now $\mathfrak{g}_\varphi \subset \mathfrak{W}_{[x]}$ gives us $\mathfrak{g}_\varphi \subset (\mathcal{P}'_x \cap \tau\mathcal{P}_x^-)$. It follows that

$$(9.19d) \quad \begin{aligned} Z \cap \mathcal{P}_x &= \mathfrak{K}' + \mathfrak{g}_\varphi + \mathfrak{g}_{-\varphi} + \mathfrak{g}_{-\tau\varphi} \quad \text{and} \\ Z \cap (\mathcal{P}_x \cap \tau\mathcal{P}_x) &= \mathfrak{K}' + \mathfrak{g}_{-\varphi} + \mathfrak{g}_{-\tau\varphi}. \end{aligned}$$

The first equation of (9.19d) says that $Z \cap \mathcal{P}_x$ is parabolic in Z , i.e. that the analytic subgroup Z of G for Z has the property that $Z(x)$ is a complex flag manifold. Let Z_0 denote the analytic subgroup of G_0 for Z_0 . Now (9.19d) gives us $\dim_{\mathbb{R}} Z(x) = 2\dim_{\mathbb{C}} Z(x) = 2\{\dim_{\mathbb{C}} Z - \dim_{\mathbb{C}}(Z \cap \mathcal{P}_x)\} = 2\{6 - 5\} = 2 = 6 - 4 = \dim_{\mathbb{C}} Z - \dim_{\mathbb{C}}(Z \cap \mathcal{P}_x \cap \tau\mathcal{P}_x) = \dim_{\mathbb{R}} Z_0 - \dim_{\mathbb{R}}(Z_0 \cap \mathcal{P}_x) = \dim_{\mathbb{R}} Z_0(x)$. Thus

$$(9.19e) \quad Z_0(x) \text{ is open in the complex flag manifold } Z(x).$$

On the other hand, (9.19a) and (9.19c) show that \mathfrak{K}'_0 is a maximally split Cartan subalgebra of Z_0 . The roots positive on $-(h_\varphi + h_{\tau\varphi}) \in \mathfrak{A}'$ are $-\varphi$ and $-\tau\varphi$. Thus we have an Iwasawa decomposition $Z_0 = \mathfrak{K}' + \mathfrak{A}' + \mathfrak{N}'$, $\mathfrak{N}' = \mathfrak{g}_{-\varphi} + \mathfrak{g}_{-\tau\varphi}$, with $(\mathfrak{A}' + \mathfrak{K}' + \mathfrak{N}') \subset Z \cap (\mathcal{P}_x \cap \tau\mathcal{P}_x)$, so Lemma 3.1 says that $Z_0(x)$ is the closed Z_0 -orbit on $Z(x)$. Combining this with (9.19e), we see that

$$Z_0 \text{ is the compact real form of } Z,$$

so the maximally compact Cartan subalgebras of Z_0 are compact. That contradicts (9.19c). We conclude that $\varphi \in A$ cannot exist, i.e. that A is empty.

Now A is empty, so $\mathfrak{N}_{[x]} = \mathfrak{W}_{[x]}$ algebra; thus $\mathfrak{N}_{[x]} = \mathfrak{N}_{[x]}$ by Theorem 8.9, so $\mathfrak{W}_{[x]} = \mathfrak{N}_{[x]}$. Thus (iii) implies (ii), completing the proof of the proposition. Q.E.D.

Let K be a maximal compact subgroup of G_0 . Then every Cartan subalgebra of \mathfrak{g}_0 is maximally split, if and only if $\text{rank } \mathfrak{g}_0 = \text{rank } \mathfrak{K} + \text{rank } G_0/K$, the latter being the symmetric space rank. The only such cases for which the symmetric space G_0/K is irreducible with G_0 acting almost-effectively are

- (9.20a) G_0 of type E_6 with K of type F_4 ,
- (9.20b) G_0 of type A_{2n-1} with K of type C_n ,
- (9.20c) G_0 of type D_{n+1} with K of type B_n , and
- (9.20d) G_0 complex simple but viewed as a real group.

9.21. COROLLARY. *Let $X = G/P$ be a complex flag manifold. Let G_0 be a real form of G that is locally isomorphic to product of compact simple*

groups and groups given by (9.20) above. Let $x \in X$. Then the following conditions are equivalent:

- (i) $G_0(x)$ is measurable.
- (ii) $\mathfrak{K}_{[x]} = (\mathcal{O}_x \cap \tau \mathcal{O}_x) + \mathcal{U}_x$.
- (iii) $(\mathcal{O}_x \cap \tau \mathcal{O}_x) + \mathcal{U}_x$ is an algebra.

PROOF. Let $\mathfrak{K} \subset \mathcal{O}_x$ be a Cartan subalgebra of \mathfrak{g} stable under τ . Then the Cartan subalgebra $\mathfrak{K}_0 = \mathfrak{K} \cap \mathfrak{g}_0$ of \mathfrak{g}_0 is a maximally split Cartan subalgebra of \mathfrak{g}_0 , by hypothesis on G_0 . Now Proposition 9.18 applies. Q.E.D.

We end §9 with a short study of the case where the complex flag manifold $X = G/P$ is a hermitian symmetric space G_u/P_u^r of compact type relative to the compact real form G_u of G , and where G'_0 is any real form of G . Here we use the notation G'_0 for purposes of reference from §10, where G_0 will denote a real form of G such that G_0/K is the noncompact hermitian symmetric space dual to $X = G_u/K$.

9.22. LEMMA. *Let $X = G/P$ be a complex flag manifold. Then the following conditions are equivalent:*

- (i) X is a hermitian symmetric space G_u/K .
- (ii) \mathcal{O}^u is abelian, i.e. $[\mathcal{O}^u, \mathcal{O}^u] = 0$.
- (iii) $[\mathcal{O}^u, \mathcal{O}^{-u}] \subset \mathcal{O}^r$.

PROOF. Let $X = G_u/K$ hermitian symmetric. Then \mathfrak{K} has a central element z that defines the almost complex structure on X by

$$\begin{aligned} \text{ad}(z) &= \sqrt{-1} \text{ on } \mathcal{O}^{-u}, & \text{ad}(z) &= -\sqrt{-1} \text{ on } \mathcal{O}^u, \\ \text{ad}(z) &= 0 \text{ on } \mathfrak{K}^c = \mathcal{O}^r. \end{aligned}$$

If $v, w \in \mathcal{O}^u$ now $\text{ad}(z)[v, w] = [z, [v, w]] = [[z, v], w] + [v, [z, w]] = [\text{ad}(z)v, w] + [v, \text{ad}(z)w] = -2\sqrt{-1}[v, w]$, while $-2\sqrt{-1}$ is not an eigenvalue of $\text{ad}(z)$; thus $[v, w] = 0$; that shows \mathcal{O}^u abelian. If $v \in \mathcal{O}^u$ and $w \in \mathcal{O}^{-u}$ then $\text{ad}(z)[v, w] = [\text{ad}(z)v, w] + [v, \text{ad}(z)w] = 0$, so $[v, w] \in \mathcal{O}^r$; that shows $[\mathcal{O}^u, \mathcal{O}^{-u}] \subset \mathcal{O}^r$. Now (i) implies (ii) and (iii).

Let $\mathfrak{K} = \mathfrak{J}^c$ be a Cartan subalgebra of \mathfrak{g} where $\mathfrak{J} \subset \mathfrak{G}_u \cap \mathcal{O}$. Choose a subset Φ of a simple \mathfrak{K} -root system Π such that $\mathcal{O} = \mathcal{O}_\Phi$. Decompose \mathfrak{g} as direct sum of simple ideals \mathfrak{L}_i , so $\mathfrak{K} = \sum \mathfrak{K}_i$ where $\mathfrak{K}_i = \mathfrak{K} \cap \mathfrak{L}_i$ is a Cartan subalgebra of \mathfrak{L}_i , and $\Pi = \cup \Pi_i$ where Π_i is a simple \mathfrak{K}_i -root system for \mathfrak{L}_i . Let μ_i be the maximal root of \mathfrak{L}_i , so $\mu_i = \sum_{\Pi_i} n_\psi \cdot \psi$, n_ψ integers ≥ 1 . If $\Pi_i \setminus \Phi$ contains a root ψ_0 with $n_{\psi_0} > 1$, we have positive roots

$$\alpha = \sum_{\Pi_i} a_\psi \cdot \psi, \quad \beta = \sum_{\Pi_i} b_\psi \cdot \psi, \quad \gamma = \alpha + \beta$$

with $a_{\psi_0} > 0$ and $b_{\psi_0} > 0$. If $\Pi_i \setminus \Phi$ contains two roots ψ_1 and ψ_2 , then we have positive roots α, β, γ as above with $a_{\psi_1} > 0$ and $b_{\psi_2} > 0$. In either case $\alpha, \beta, \gamma \in \Phi^u$, so

$\mathfrak{G}_\gamma = [\mathfrak{G}_\alpha, \mathfrak{G}_\beta] \subset [\mathfrak{O}^u, \mathfrak{O}^u]$ shows that (ii) fails, and

$\mathfrak{G}_\alpha = [\mathfrak{G}_\gamma, \mathfrak{G}_{-\beta}] \subset [\mathfrak{O}^u, \mathfrak{O}^{-u}] \cap \mathfrak{O}^u$ shows that (iii) fails.

Thus (ii) and (iii) each implies that, for every i , either $\Pi_i \subset \Phi$ or $\Pi_i \setminus \Phi = \{\psi_0\}$ with $n_{\psi_0} = 1$. It is immediate [10, Lemma 3] that $G_u/G_u \cap P = G_u/P'_u$ is a realization of X as a hermitian symmetric space of compact type. Q.E.D.

9.23. REMARK. Let $X = G/P$ be a complex flag manifold that is hermitian symmetric. Let G'_0 be any real form of G , τ' conjugation of \mathfrak{g} over \mathfrak{g}'_0 . If $x \in X$ then Lemma 9.22 shows that $\mathfrak{W}_{[x]} = (\mathfrak{O}_x \cap \tau' \mathfrak{O}_x) + (\mathfrak{O}_x^u \cap \tau' \mathfrak{O}_x^{-u}) + (\mathfrak{O}_x^{-u} \cap \tau' \mathfrak{O}_x^u)$ is a subalgebra of \mathfrak{g} . As

$$(\mathfrak{O}_x \cap \tau' \mathfrak{O}_x) \subset \mathfrak{W}_{[x]} = \tau' \mathfrak{W}_{[x]} \subset (\mathfrak{O}_x + \tau' \mathfrak{O}_x),$$

Lemma 8.4 says that $\mathfrak{W}_{[x]} \subset \mathfrak{X}_{[x]}$, so Theorem 9.7 says that $G'_0(x)$ is of flag type. However, Example 8.12 shows that one cannot expect $G'_0(x)$ to be partially complex without further restriction on X, x , or G'_0 . In that context we prove

9.24. PROPOSITION. *Let $X = G/P$ be a complex flag manifold that is a hermitian symmetric space. Let G'_0 be any real form of G . Decompose*

$$\mathfrak{g} = \mathfrak{g}^1 \oplus \dots \oplus \mathfrak{g}^r, \quad \mathfrak{O} = \mathfrak{O}^1 \oplus \dots \oplus \mathfrak{O}^r, \quad X = X^1 \times \dots \times X^r$$

where the \mathfrak{g}^i are the minimal τ' -stable ideals of \mathfrak{g} , $\mathfrak{O}^i = \mathfrak{O} \cap \mathfrak{g}^i$, and $X^i = G^i/P^i$ quotient of the analytic subgroups of G for \mathfrak{g}^i and \mathfrak{O}^i ; so the $\mathfrak{g}'_0{}^i = \mathfrak{g}'_0 \cap \mathfrak{g}^i$ are the simple ideals of \mathfrak{g}'_0 . Then the following conditions are equivalent:

(1) *Some open G'_0 -orbit on X is a symmetric coset space of G'_0 , with invariant nondegenerate kaehler metric that may be of indefinite signature.*

(1') *Every open G'_0 -orbit on X is a symmetric coset space of G'_0 , with invariant nondegenerate kaehler metric that may be of indefinite signature.*

(2) *Some open G'_0 -orbit on X is measurable.*

(2') *Every open G'_0 -orbit on X is measurable.*

(3) *For each index $i, 0 \leq i \leq r$, if \mathfrak{g}^i is simple then either*

(3a) *the maximal compactly embedded subalgebras $\mathfrak{K}^i \subset \mathfrak{g}'_0{}^i$ satisfy rank $\mathfrak{K}^i = \text{rank } \mathfrak{g}'_0{}^i$, or*

(3b) *X^i is the grassmannian $\mathbf{U}(2n)/\mathbf{U}(n) \times \mathbf{U}(n)$ of n -planes through the origin in \mathbf{C}^{2n} with $\mathfrak{g}^i = \mathfrak{sl}(2n, \mathbf{C})$ and either $\mathfrak{g}'_0{}^i = \mathfrak{sl}(2n, \mathbf{R})$ or $\mathfrak{g}'_0{}^i = \mathfrak{sl}(n, \mathbf{Q})$, or*

(3c) *X^i is the complex quadric $\mathbf{SO}(2n+2)/\mathbf{SO}(2n) \times \mathbf{SO}(2)$ with $\mathfrak{g}^i = \mathfrak{so}(2n+2, \mathbf{C})$ and $\mathfrak{g}'_0{}^i = \mathfrak{so}^{2t+1}(2n+2)$ where $0 \leq t \leq [n/2]$.*

(4) For each index i , \mathfrak{G}^i simple, either

(4a) $\text{rank } \mathfrak{K}^i = \text{rank } \mathfrak{G}_0^i$, or

(4b) the noncompact dual of the irreducible compact hermitian symmetric space X^i is of tube type, but not isomorphic to an $\text{SO}^*(2l)/\text{U}(l)$ for $l > 4$.

PROOF. Let $x \in X$ such that $G'_0(x)$ is open in X . If $G'_0(x)$ is symmetric, say with symmetry σ at x , then the isotropy subalgebra $\mathfrak{G}'_0 \cap \mathcal{P}_x = \{v \in \mathfrak{G}'_0 : \sigma(v) = v\}$ is reductive in \mathfrak{G}'_0 , so Lemma 6.2 says that $G'_0(x)$ is measurable. If $G'_0(x)$ is measurable, Theorem 6.3 says

(i) there is an invariant, possibly indefinite, kaehler metric on $G'_0(x)$ and

(ii) $\tau' \mathcal{P}_x^r = \mathcal{P}_x^r$ and $\tau' \mathcal{P}_x^u = \mathcal{P}_x^{-u}$.

Choose a compact real form G_u of G such that \mathfrak{G}'_0 is invariant under the automorphism σ of \mathfrak{g} induced from conjugation of \mathfrak{G}_u by the symmetry at x ; σ is $+1$ on $\mathfrak{G}_u \cap \mathcal{P}_x = (\mathcal{P}_x^r)_u$ and -1 on a complement; thus σ is $+1$ on \mathcal{P}_x^r and -1 on $\mathcal{P}_x^u + \mathcal{P}_x^{-u}$; now $\mathfrak{G}'_0 \cap \mathcal{P}_x = (\text{real form of } \mathcal{P}_x^r) = \{v \in \mathfrak{G}'_0 : \sigma(v) = v\}$, so $G'_0(x)$ is symmetric. We have proved equivalence of (1) and (2) for $G'_0(x)$. As (2) and (2') are equivalent by Theorem 6.7, we now have equivalence of (1), (1'), (2) and (2').

In proving (2), (3) and (4) equivalent, it suffices to consider the case where \mathfrak{G}'_0 is simple. For if $G_0(x)$ is open in X , and $x = (x^1, \dots, x^r)$ with $x^i \in X^i$, then Theorem 6.7 shows that $G'_0(x)$ is measurable if and only if each $G_0^i(x^i)$ is measurable. So we now assume that \mathfrak{G}'_0 is simple, and we fix $x \in X$ with $G'_0(x)$ open in X .

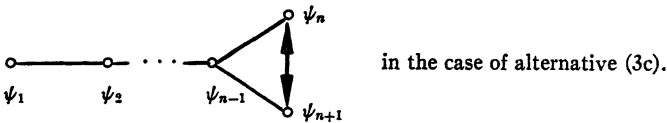
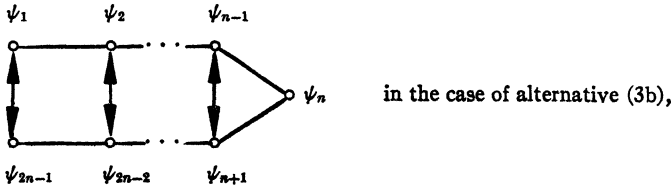
If \mathfrak{g} is not simple, i.e. if \mathfrak{G}'_0 is not absolutely simple, then \mathfrak{G}'_0 is the real structure of a complex simple Lie algebra \mathfrak{L} , $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{L}$ with \mathfrak{G}'_0 embedded diagonally, and τ' acts on \mathfrak{g} by $(u, v) \rightarrow (v, u)$. Lemma 9.22 shows that $(\mathcal{P}_x \cap \tau' \mathcal{P}_x) + (\mathcal{P}_x^u \cap \tau' \mathcal{P}_x^{-u}) + (\mathcal{P}_x^{-u} \cap \tau' \mathcal{P}_x^u)$ is an algebra; that is condition (iii) of Corollary 9.21; thus $G'_0(x)$ is measurable.

Now let \mathfrak{g} be simple. If the maximal compactly embedded subalgebras $\mathfrak{K} \subset \mathfrak{G}'_0$ satisfy $\text{rank } \mathfrak{K} = \text{rank } \mathfrak{G}'_0$, then Corollary 6.4 says that $G'_0(x)$ is measurable.

Finally suppose \mathfrak{g} simple with $\text{rank } \mathfrak{K} < \text{rank } \mathfrak{G}'_0$. If $G'_0(x)$ is measurable, we have the symmetry σ of $X = G_u/P'_u$ at x , such that $\sigma \mathfrak{G}'_0 = \mathfrak{G}'_0$ and $\mathfrak{G}'_0 \cap \mathcal{P}_x = \{v \in \mathfrak{G}'_0 : \sigma(v) = v\}$, real form of \mathcal{P}_x^r . Let α be a Cartan involution of \mathfrak{G}'_0 that commutes with σ , $\mathfrak{G}'_0 = \mathfrak{K} + \mathfrak{M}_0$ the Cartan decomposition under α . Then $\mathfrak{G}_u = \mathfrak{K} + \sqrt{-1} \mathfrak{M}_0$, and $\text{rank } \mathfrak{K} < \text{rank } \mathfrak{G}_u$, so α carries over to an involutive outer automorphism of \mathfrak{G}_u that preserves $(\mathcal{P}_x^r)_u$, thus acts on $X = G_u/(P'_x)_u$, and preserves the invariant complex structure on X . We apply [11, Theorem 13.3] to α and $X = G_u/(P'_x)_u$, concluding that either $X = \text{SU}(2n)/\text{S}(\text{U}(n) \times \text{U}(n))$ with α interchanging the two $\text{U}(n)$ factors, or $X = \text{SO}(2n+2)/\text{SO}(2n)$

$\times \text{SO}(2)$ with α acting on $\text{SO}(2n) \times \text{SO}(2) \subset \text{O}(2n) \times \text{SO}(2)$ as conjugation by some (Q, P) with $\det Q = -1$. The other cases of [11, Theorem 13.3(2)] are eliminated because they are not symmetric. Thus measurability of $G'_0(x)$ implies alternative (3b) or (3c).

Conversely, suppose that we have alternative (3b) or (3c). Then $\text{rank } \mathcal{K} < \text{rank } \mathfrak{g}'_0$ shows that τ' induces an outer automorphism of a compact real form \mathfrak{g}_u of \mathfrak{g} . Let \mathfrak{J} denote a τ' -invariant Cartan subalgebra of \mathfrak{g}_u . Choose a τ' -invariant Weyl chamber for the Cartan subalgebra $\mathcal{K} = \mathfrak{J}^\mathcal{C}$ of \mathfrak{g} . Then τ' acts on the corresponding simple root system Π by



On the other hand, \mathcal{O} is conjugate to \mathcal{O}_Φ where, in the notation just above,

$$\Phi = \{\psi_1, \dots, \psi_{n-1}; \psi_{n+1}, \dots, \psi_{2n-1}\} \text{ in alternative (3b),}$$

$$\Phi = \{\psi_2, \psi_3, \dots, \psi_{n-1}, \psi_n, \psi_{n+1}\} \text{ in alternative (3c).}$$

Thus $\tau'\Phi = \Phi$, so $\tau'\mathcal{O}_\Phi = \mathcal{O}_\Phi$. But \mathcal{O}_Φ is opposed to itself. Thus \mathcal{O} and $\tau'\mathcal{O}$ are opposed. Theorem 6.7 now says that $G'_0(x)$ is measurable.

We have proved that (2) and (3) are equivalent.

Suppose \mathfrak{g} simple with $\text{rank } \mathcal{K} < \text{rank } \mathfrak{g}'_0$. Then \mathfrak{g} has an involutive outer automorphism, so it is of type A_l, D_l or E_6 . Let X_0 denote the noncompact dual of $X = G_u/P'_u$. If \mathfrak{g} is of type A_l , then X is the grassmannian of k -planes through the origin in \mathbf{C}^{l+1} , $2k \leq l+1$, and X_0 is of tube type if and only if $2k = l+1$. If \mathfrak{g} is of type D_l , then either $X = \text{SO}(2l)/\text{U}(l)$ where X_0 is of tube type if and only if l is even, or $X = \text{SO}(2l)/\text{SO}(2l-2) \times \text{SO}(2)$ complex quadric with X_0 of tube type. If \mathfrak{g} is of type E_6 then X_0 is not of tube type. That proves equivalence of (3) and (4), completing the proof of the proposition. Q.E.D.

9.25. COROLLARY. *Let $X = G/P$ be a complex flag manifold that is a*

hermitian symmetric space. Let G'_0 be any real form of G such that the maximal compact subgroups $K \subset G'_0$ satisfy $\text{rank } K = \text{rank } G'_0$. Then every open G'_0 -orbit on X is a symmetric coset space of G'_0 with invariant, nondegenerate, possibly indefinite, kaehler metric.

For that is a special case of alternative (3a) of Proposition 9.24.

CHAPTER III. HERMITIAN SYMMETRIC SPACES

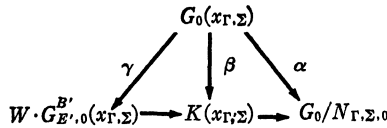
Let $X = G/P$ be a complex flag manifold that is a hermitian symmetric space of compact type. In other words $X = G_u/K$, where G_u is the compact real form of G and $K = P \cap G_u$ (necessarily a compact real form of P) is a symmetric subgroup of G_u . Let G_0 denote the real form of G such that $X_0 = G_0/K$ is the hermitian symmetric space of noncompact type dual to X . §10 is a complete analysis of the G_0 -orbit structure of X and the holomorphic arc component structure of an arbitrary G_0 -orbit. §11 carries that analysis over to a decomposition theory for an arbitrary orbit $G_0(x)$ that extends the Siegel domain theory for the convex open orbit X_0 . A summary of the Siegel domain theory for X_0 and its reformulation as a theory of holomorphic fibrations of X_0 over its boundary components, is collected in the Appendix to §11 for use in our decomposition of $G_0(x)$.

In §10 we pick out certain points $x_{\Gamma, \Sigma} \in X$ by means of the Wolf-Korányi partial Cayley transform theory [15], and certain complex subgroups $G_{\Psi \setminus \Gamma} \subset G$ and their G_0 -real forms $G_{\Psi \setminus \Gamma, 0}$, such that the G_0 -orbits on X are just the $G_0(x_{\Gamma, \Sigma})$ and the holomorphic arc components of $G_0(x_{\Gamma, \Sigma})$ are just the $k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$ where $k \in K$. More precisely let Ψ be a maximal set of strongly orthogonal noncompact positive roots of \mathfrak{g} relative to a Cartan subalgebra \mathfrak{J}^c with $\mathfrak{J} \subset \mathfrak{K}$. Any two choices of (\mathfrak{J}, Ψ) are $\text{ad}_G(K)$ -conjugate, $|\Psi| = \text{rank } X_0$, and $X_0 = K \cdot G_0[\Psi](x_0)$ where x_0 is the base point and $G_0[\Psi](x_0) = \prod_{\Psi} G_0[\psi](x_0)$ polydisc in X_0 . If $\psi \in \Psi$ then c_ψ is the Cayley transform on the Riemann sphere $G[\psi](x_0)$, rotation by $\pi/2$ carrying the south pole x_0 up to the equator. If $B \subset \Psi$ then $c_B = \prod_B c_\psi$. If $\Gamma, \Sigma \subset \Psi$ then $x_{\Gamma, \Sigma} = c_{\Gamma \cup \Sigma} x_0$ and $G_{\Psi \setminus \Gamma}$ is the derived group of the centralizer of $\mathfrak{J}^c + \sum_{\psi \perp \Gamma} \mathfrak{g}_\psi$ in G .

Theorem 10.6 says that the G_0 -orbits on X are the $G_0(x_{\Gamma, \Sigma})$, Γ and Σ disjoint in Ψ , and gives criteria for $G_0(x_{\Gamma', \Sigma'}) = G_0(x_{\Gamma, \Sigma})$ and $G_0(x_{\Gamma', \Sigma'}) \subset \overline{G_0(x_{\Gamma, \Sigma})}$. If $\{r_1, \dots, r_p\}$ are the ranks of the irreducible factors of X_0 , it follows that there are precisely $\frac{1}{2} \sum_{1 \leq i \leq p} (r_i + 1)(r_i + 2)$ G_0 -orbits on X . The open G_0 -orbits are the $G_0(x_{\phi, \Sigma})$; they are $\sum_{1 \leq j \leq p} (r_j + 1) = |\Psi| + p$ in number, are indefinite-hermitian symmetric spaces of G_0 , and are indefinite metric versions of X_0 . Theorem

10.9 says that $G_{\Psi \setminus \Gamma}(x_{\Gamma, \Sigma}) = G_{\Psi \setminus \Gamma}(x_{\Gamma, \phi})$ is a compact totally geodesic hermitian symmetric subspace of rank $|\Psi \setminus \Gamma|$ in X , that the $G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$, $\Gamma \cap \Sigma = \phi$, are the open $G_{\Psi \setminus \Gamma, 0}$ -orbits on it, that the holomorphic arc components of $G_0(x_{\Gamma, \Sigma})$ are the $k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$ with $k \in K$, and that the complex Lie algebra normalizer $\mathfrak{N}_{[x_{\Gamma, \Sigma}]} = \mathfrak{N}_{\Gamma, \Sigma}$ of $G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$ depends only on Γ . The method is rather delicate; Theorem 8.9 is used to show $\mathfrak{N}_{\Gamma, \Sigma}$ independent of Σ , and results from Wolf-Korányi [15] are used to determine $\mathfrak{N}_{\Gamma, \phi}$. In particular every holomorphic arc component of every G_0 -orbit is indefinite-hermitian symmetric, so every G_0 -orbit is measurable.

In §11 we work out a decomposition:



There α is the K -equivariant fibration whose fibres are the holomorphic arc components of $G_0(x_{\Gamma, \Sigma})$. β is a K -equivariant fibration whose restriction to a holomorphic arc component is a holomorphic fibration over the maximal compact subvariety of that component; the fibres of β are hermitian symmetric spaces $k c_{\Gamma} c_{\Sigma}^2 \{ G_{\Psi \setminus (\Gamma \cup \Sigma), 0}(x_0) \times G_{\Sigma, 0}^2(x_0) \}$ of noncompact type whose second factor is the tube part of $G_{\Sigma, 0}(x_0)$. And γ is an a.e. analytic measurable bundle whose restriction to $c_{\Gamma} c_{\Sigma}^2 \{ G_{\Psi \setminus (\Gamma \cup \Sigma), 0}(x_0) \times G_{\Sigma, 0}^2(x_0) \}$ is a holomorphic fibration over a bounded symmetric domain, any of the Siegel domain realizations of type III for a boundary component of $G_{\Psi \setminus (\Gamma \cup \Sigma), 0}(x_0) \times G_{\Sigma, 0}^2(x_0)$. The Siegel domain theory used in γ is in the Appendix to §11. The fibration β plays two roles. It gets rid of compact subvarieties of holomorphic arc components so that some function theory is possible despite the injunction of Theorem 5.7, and it allows construction of the "generalized Siegel domain" γ in a manner whose discontinuities do not interfere with integration.

NOTES FOR CHAPTER III. The results of Chapter III depend in an essential manner, both on the results of Chapter II, and on the earlier work of Wolf-Korányi [15]. The latter contains the theorems of §10 for those G_0 -orbits that are in the topological closure of X_0 .

Enumeration of the G_0 -orbits on X (Theorem 10.6) was first worked out by Takeuchi [9]. Using the partial Cayley transforms of Wolf-Korányi [15] it is not difficult to see that every G_0 -orbit must be one of the $G_0(x_{\Gamma, \Sigma})$, so the problem is reduced to that of finding a criterion for $G_0(x_{\Gamma', \Sigma'}) = G_0(x_{\Gamma, \Sigma})$. There, a little serious thought reduces the question to the case $\Gamma' = \phi = \Gamma$ of open orbits. Takeuchi [9] settles the

latter by picking out the maximal compact subvarieties of open orbits and counting them with the aid of some Morse theory results on cell decompositions from his thesis [8]. We find it more convenient to apply Theorem 4.9 and count some double cosets.

The fibration $\beta: G_0(x_{\Gamma, \Sigma}) \rightarrow K(x_{\Gamma, \Sigma})$ was worked out by Takeuchi [9] for the case $\Gamma = \phi$ of open orbits, where holomorphic arc components are not needed; part 3 of Theorem 11.8 answers (in general) the triviality question on β that Takeuchi raised in his special case. A key to our result Theorem 11.8 on β is Lemma 11.6, which was proved by Takeuchi [9], for the case $\Gamma = \phi$ of open orbits, in a different way.

Except for the overlap with Wolf-Korányi [15] and Takeuchi [9] described above, the material of §10 and §11 was worked out by this author in the fall and winter of 1968.

10. Hermitian symmetric spaces: orbit structure and holomorphic arc components. Let $X = G_u/\mathcal{K}$ be a hermitian symmetric space of compact type. Then X is a flag manifold G/P where G_u is the compact real form of G and $K = P \cap G_u$ is the compact real form of P^r . Now let $X_0 = G_0/K$ be the dual hermitian symmetric space of noncompact type; if $\mathfrak{g}_u = \mathcal{K} + \mathfrak{M}_u$ Cartan decomposition, then $\mathfrak{g}_0 = \mathcal{K} + \mathfrak{M}_0$, $\mathfrak{M}_0 = \sqrt{-1}\mathfrak{M}_u$, real form of \mathfrak{g} such that $K = P \cap G_0$, maximal compact subgroup of G_0 . The "Borel embedding" of X_0 in X is the embedding $b: X_0 \rightarrow X$ of X_0 as an open G_0 -orbit on X , given by $b(gK) = gP$ for $g \in G_0$. Let x_0 denote the identity coset in $X = G/P$, so $b(X_0) = G_0(x_0)$. We map \mathcal{O}^{-u} onto a dense open subset of X by $\xi(z) = \exp(z) \cdot x_0$. Then the "Harish-Chandra embedding" of X_0 as a bounded domain D in a complex euclidean space \mathcal{O}^{-u} is given by $D = \kappa(X_0)$ where $\kappa(gK) = \xi^{-1}b(gK)$, and $\xi: D \cong G_0(x_0)$ relates the two embeddings. One views the inclusions $\{\xi(D) = G_0(x_0)\} \subset \xi(\mathcal{O}^{-u}) \subset X$ as a generalization of the classical (unit disc) \subset (complex line) \subset (Riemann sphere).

Bott-Korányi (see [2, Theorem 3.6]) used convexity properties of D in \mathcal{O}^{-u} to identify the Bergman-Šilov boundary of $G_0(x_0)$ in X as the unique closed G_0 -orbit in the topological boundary $\partial G_0(x_0)$ of $G_0(x_0)$ in X . Korányi-Wolf [2] studied the structure of the Bergman-Šilov boundary and its role in the realization of X_0 as a Siegel Domain of type II. Then Wolf-Korányi [15] made full use of the convexity of D in \mathcal{O}^{-u} to obtain the G_0 -orbit structure of $\partial G_0(x_0)$, the holomorphic arc components of $\partial G_0(x_0)$ (there called the "boundary components" of $G_0(x_0)$ in X), and the realizations of X_0 as Siegel Domains of type III. There, it was shown that every holomorphic arc component of $\partial G_0(x_0)$ lay in a boundary orbit; so the holomorphic arc components of $\partial G_0(x)$ are the same as the holomorphic arc components of orbits $G_0(x)$, $x \in \partial G_0(x_0)$. Having seen Korányi-Wolf [2] but not Wolf-

Korányi [15], Maebashi [6] enumerated the G_0 -orbits on X for the case where X_0 is the Siegel upper half plane $\mathbf{Sp}(n, \mathbf{R})/\mathbf{U}(n)$. Finally, Takeuchi [3] extended part of Wolf-Korányi [15], enumerating the G_0 -orbits on X in general by means of a recursive procedure based on a counting of the open G_0 -orbits; the latter consisted of describing a set consisting of just one maximal compact subvariety from each open G_0 -orbit as the critical set of a certain function on X . Nonconvexity of the general open G_0 -orbit on X prevented Takeuchi from extending the boundary component theory. That difficulty is avoided by our notion of holomorphic arc component developed in §§8 and 9, and this allows us to give a complete treatment of the G_0 -orbit structure of X .

We establish notation for hermitian symmetric spaces.

X_0 : hermitian symmetric space of noncompact type.

G_0 : largest connected group of isometries of X_0 .

x_0 : fixed "base point" in X_0 .

K : isotropy subgroup of G_0 at x_0 , maximal compact in G_0 .

σ : Cartan involution $\text{ad}(s)$, where s is the symmetry of X_0 at x_0 .

Let $\mathfrak{K} \subset \mathfrak{g}_0$ denote the Lie algebras of $K \subset G_0$. Then

$\mathfrak{g}_0 = \mathfrak{K} + \mathfrak{M}_0$: Cartan decomposition into (± 1) -eigenspaces of σ .

$\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}} = \mathfrak{K}^{\mathbb{C}} + \mathfrak{M}$ complexification where $\mathfrak{M} = \mathfrak{M}_0^{\mathbb{C}}$.

$\mathfrak{g}_u = \mathfrak{K} + \mathfrak{M}_u$, $\mathfrak{M}_u = i\mathfrak{M}_0$, compact real form of \mathfrak{g} .

G : adjoint group of \mathfrak{g} , so G_0 is the analytic subgroup for \mathfrak{g}_0 .

G_u : analytic subgroup of G for \mathfrak{g}_u , compact real form of G .

X : G_u/K , hermitian symmetric space, compact dual of X_0 .

\mathfrak{J} : Cartan subalgebra of \mathfrak{g}_0 (or \mathfrak{g}_u) contained in \mathfrak{K} .

Δ : $\mathfrak{J}^{\mathbb{C}}$ -root system of \mathfrak{g} .

Δ_K : compact roots, i.e. $\mathfrak{J}^{\mathbb{C}}$ -root system of $\mathfrak{K}^{\mathbb{C}}$.

Δ_M : noncompact roots, i.e. roots φ such that $\mathfrak{g}_{\varphi} \subset \mathfrak{M}$.

z : central element of \mathfrak{K} such that $J = \text{ad}(z)|_{\mathfrak{M}}$ is the almost complex structure of X_0 and X .

\mathfrak{M}^{\pm} : $(\mp \sqrt{-1})$ -eigenspaces of $J = \text{ad}(z)|_{\mathfrak{M}}$.

\mathcal{O} : $\mathfrak{K}^{\mathbb{C}} + \mathfrak{M}^+$, parabolic subalgebra of \mathfrak{g} defined by iz .

P : parabolic subgroup of G with Lie algebra \mathcal{O} .

We observe $\mathcal{O}^r = \mathfrak{K}^{\mathbb{C}}$, $\mathcal{O}^u = \mathfrak{M}^+$ and $\mathcal{O}^{-u} = \mathfrak{M}^-$. Theorem 5.4 says $G_0 \cap P = K = G_u \cap P$. Thus G_u is transitive on the complex flag manifold G/P with isotropy subgroup K . That identifies $X = G_u/K$ with the complex flag manifold G/P , providing the Borel embedding $b: X_0 \rightarrow X$ by $b(gK) = gP$ for $g \in G_0$. Now we identify X_0 with $b(X_0)$, thus identifying the base point x_0 with the identity coset of X , thus identifying X_0 with the open orbit $G_0(x_0)$.

Choose a simple root system Π such that $\mathcal{P} = \mathcal{P}_\Phi$ for some subset $\Phi \subset \Pi$. Then Φ consists of the compact simple roots and $\Pi \setminus \Phi$ the noncompact simple roots. There is one noncompact simple root for each simple ideal of \mathfrak{g} . Note

$$\Phi^r = \Delta_K, \quad \Phi^u = \Delta_M^+, \quad -\Phi^u = \Delta_M^-.$$

Two roots φ and ψ are *strongly orthogonal* if neither of $\varphi \pm \psi$ is a root; in that case $\langle \varphi, \psi \rangle = 0$. Now we define

Ψ : maximal set of strongly orthogonal positive noncompact roots by $\Psi = \{\Psi_1, \dots, \Psi_r\}$ where Ψ_{j+1} is the lowest element of Δ_M^+ strongly orthogonal to each of $\{\psi_1, \dots, \psi_j\}$. Thus, for each simple ideal of \mathfrak{g} , the noncompact simple root is the first element of Ψ that is a root of the ideal. The number $r = |\Psi|$ is the common symmetric space rank of X_0 and X . Define a real basis of \mathfrak{M}_0 consisting of all

$$x_{\varphi,0} = e_\varphi + e_{-\varphi} \quad \text{and} \quad y_{\varphi,0} = -i(e_\varphi - e_{-\varphi}), \varphi \in \Delta_M^+,$$

where $e_{\pm\varphi} \in \mathfrak{g}_{\pm\varphi}$ are normalized by $[e_\varphi, e_{-\varphi}] = h_\varphi \in i\mathfrak{J}$. Then \mathfrak{M}_u has a real basis consisting of all

$$x_\varphi = i(e_\varphi + e_{-\varphi}) \quad \text{and} \quad y_\varphi = e_\varphi - e_{-\varphi}, \quad \varphi \in \Delta_M^+.$$

That gives us

$$\mathfrak{A}_0: \sum_{\Psi} \mathbf{R}x_{\psi,0}, \quad \text{maximal abelian subspace of } \mathfrak{M}_0.$$

$$\mathfrak{A}_u: i\mathfrak{A}_0 = \sum_{\Psi} \mathbf{R}x_\psi, \quad \text{maximal abelian subspace of } \mathfrak{M}_u.$$

For every noncompact positive root φ we compute

$$Jx_{\varphi,0} = [z, x_{\varphi,0}] = y_{\varphi,0}, \quad Jy_{\varphi,0} = [z, y_{\varphi,0}] = -x_{\varphi,0}, \quad [x_{\varphi,0}, y_{\varphi,0}] = 2ih_\varphi,$$

$$Jx_\varphi = [z, x_\varphi] = y_\varphi, \quad Jy_\varphi = [z, y_\varphi] = -x_\varphi, \quad [x_\varphi, y_\varphi] = -2ih_\varphi.$$

In particular, using strong orthogonality of the roots in Ψ ,

$$[\mathfrak{A}_0, J\mathfrak{A}_0] = [\mathfrak{A}_u, J\mathfrak{A}_u] = \sum_{\Psi} (ih_\psi)\mathbf{R}.$$

That gives

(10.1a) $\mathfrak{J} = \mathfrak{J}^+ + \mathfrak{J}^-$ orthogonal direct sum, where

(10.1b) $\mathfrak{J}^- = [\mathfrak{A}_0, J\mathfrak{A}_0]$ and \mathfrak{J}^+ is the centralizer of \mathfrak{A}_0 in \mathfrak{J} .

For every subset $\Gamma \subset \Psi$ we define

(10.2a) $\mathfrak{J}_\Gamma^- = \sum_{\Psi \setminus \Gamma} (ih_\psi)\mathbf{R}$, so $\mathfrak{J}_\emptyset^- = \mathfrak{J}^-$ and $\mathfrak{J}_\Psi^- = 0$;

(10.2b) $\mathfrak{A}_\Gamma = \sum_{\Gamma} (x_{\gamma,0})\mathbf{R}$, so $\mathfrak{A}_\emptyset = 0$ and $\mathfrak{A}_\Psi = \mathfrak{A}_0$;

(10.2c) $\mathfrak{K}_\Gamma = \mathfrak{J}^+ + \mathfrak{J}_\Gamma^- + \mathfrak{A}_\Gamma$, so $\mathfrak{K}_\emptyset = \mathfrak{J}$ and $\mathfrak{K}_\Psi = \mathfrak{J}^+ + \mathfrak{A}_0$.

Note that (10.2c) is an orthogonal direct sum. The \mathfrak{K}_Γ are the Cartan subalgebras of \mathfrak{G}_0 with which we shall work. They are related by the partial Cayley transforms [15] defined by

(10.3a) $c_\psi = \exp((\pi/4)y_\psi) \in G_u$ for every $\psi \in \Psi$, and

(10.3b) $c_\Gamma = \prod_{\Gamma} c_\gamma \in G_u$ for every $\Gamma \subset \Psi$.

Now we have

(10.4a) $\text{ad}(c_\psi): y_\psi \rightarrow y_\psi, x_\psi \rightarrow 2ih_\psi, 2ih_\psi \rightarrow -x_\psi$; so

(10.4b) $\text{ad}(c_{\Gamma' \setminus \Gamma})\mathfrak{K}_\Gamma^c = \mathfrak{K}_{\Gamma'}^c$ for $\Gamma \subset \Gamma' \subset \Psi$, and

(10.4c) $\text{ad}(c_\Sigma)^2\mathfrak{K}_\Gamma = \mathfrak{K}_\Gamma$ for $\Gamma, \Sigma \subset \Psi$.

We use the partial Cayley transforms to define points

(10.5) $x_{\Gamma, \Sigma} = c_{\Gamma} c_{\Sigma}^2 x_0 \in X$ whenever $\Gamma, \Sigma \subset \Psi$.

Now we can state the decomposition of X into G_0 -orbits.

10.6 THEOREM. *Decompose $\mathfrak{G} = \mathfrak{G}^1 \oplus \dots \oplus \mathfrak{G}^p$ direct sum of simple ideals, so $\Delta = \Delta^1 \cup \dots \cup \Delta^p$ where Δ^k is the root system of \mathfrak{G}^k .*

1. *The G_0 -orbits on X are just the $G_0(x_{\Gamma, \Sigma})$, Γ and Σ disjoint.*
2. *Orbits $G_0(x_{\Gamma, \Sigma})$ and $G_0(x_{\Gamma', \Sigma'})$ are equal if and only if, for $1 \leq k \leq p$, $|\Sigma \setminus \Gamma' \cap \Delta^k| = |\Sigma \setminus \Gamma \cap \Delta^k|$ and $|\Gamma' \cap \Delta^k| = |\Gamma \cap \Delta^k|$. In particular, the number of G_0 -orbits on X is $\frac{1}{2} \sum_{1 \leq k \leq p} (r_k + 1)(r_k + 2)$ where $r_k = |\Psi \cap \Delta^k|$.*
3. *$G_0(x_{\Gamma', \Sigma'})$ is in the closure of $G_0(x_{\Gamma, \Sigma})$ if and only if, for $1 \leq k \leq p$, $|\Sigma \setminus \Gamma' \cap \Delta^k| \leq |\Sigma \setminus \Gamma \cap \Delta^k|$ and $|\Sigma \cup \Gamma \cap \Delta^k| \leq |\Sigma \cup \Gamma' \cap \Delta^k|$.*
4. *$G_0(x_{\Gamma, \Sigma})$ is open in X if and only if $\Gamma = \phi$ empty set. In particular there are precisely $\sum_{1 \leq k \leq p} (r_k + 1) = |\Psi| + p$ open G_0 -orbits on X .*
5. *$G_0(x_{\Psi, \phi})$ is the closed G_0 -orbit on X , Bergman-Silov boundary of X_0 in X , contained in the closure of every G_0 -orbit.*

The corresponding result for Cartan subalgebras is

10.7. THEOREM. *Let \mathfrak{G}^k and Δ^k be as in Theorem 10.6.*

1. *\mathfrak{K}_Γ is maximally compact among the Cartan subalgebras of \mathfrak{G}_0 contained in $\mathfrak{O}_{x_{\Gamma, \Sigma}}$, $\Gamma \cap \Sigma = \phi$, and every Cartan subalgebra of \mathfrak{G}_0 is conjugate to some \mathfrak{K}_Γ .*
2. *\mathfrak{K}_Γ and $\mathfrak{K}_{\Gamma'}$ are conjugate in \mathfrak{G}_0 , if and only if, for $1 \leq k \leq p$, $|\Gamma \cap \Delta^k| = |\Gamma' \cap \Delta^k|$.*
3. *\mathfrak{K}_Γ centralizes a \mathfrak{G}_0 -conjugate of $\mathfrak{J}_{\Gamma'}$, if and only if, for $1 \leq k \leq p$, $|\Gamma \cap \Delta^k| \leq |\Gamma' \cap \Delta^k|$.*
4. *\mathfrak{K}_Γ is maximally compact in \mathfrak{G}_0 if and only if $\Gamma = \phi$.*
5. *\mathfrak{K}_Γ is maximally split in \mathfrak{G}_0 if and only if $\Gamma = \Psi$.*

We will prove Theorems 10.6 and 10.7 as we decompose a G_0 -orbit into holomorphic arc components. To express that decomposition, note that

$$\mathfrak{J}^c + \sum_{\varphi \in \Gamma} \mathfrak{G}_\varphi \text{ is the centralizer of } \mathfrak{J}_{\Psi \setminus \Gamma} \text{ in } \mathfrak{g}.$$

Now define

(10.8a) $\mathfrak{G}_{\Psi \setminus \Gamma}$ is the derived algebra of $\mathfrak{I}^\sigma + \sum_{\varphi \perp \Gamma} \mathfrak{G}_\varphi$,

(10.8b) $\mathfrak{G}_{\Psi \setminus \Gamma, 0}$ is the real form $\mathfrak{G}_0 \cap \mathfrak{G}_{\Psi \setminus \Gamma}$ of $\mathfrak{G}_{\Psi \setminus \Gamma}$, and

(10.8c) $G_{\Psi \setminus \Gamma}$ and $G_{\Psi \setminus \Gamma, 0}$ are the analytic subgroups for $\mathfrak{G}_{\Psi \setminus \Gamma}$ and $\mathfrak{G}_{\Psi \setminus \Gamma, 0}$.

10.9. THEOREM. *Let Γ and Σ be arbitrary subsets of Ψ .*

1. $G_{\Psi \setminus \Gamma}(x_{\Gamma, \Sigma}) = c_{\Gamma} c_{\Sigma}^2 G_{\Psi \setminus \Gamma}(x_0)$, compact totally geodesic hermitian symmetric subspace of X ; its symmetric space rank is $|\Psi \setminus \Gamma|$; relative to the Cartan subalgebra $\mathfrak{I} \cap \mathfrak{G}_{\Psi \setminus \Gamma, 0}$, $\Psi \setminus \Gamma$ is a maximal set of strongly orthogonal noncompact roots of $\mathfrak{G}_{\Psi \setminus \Gamma, 0}$.

2. $G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$ is an open $G_{\Psi \setminus \Gamma, 0}$ -orbit on the symmetric complex flag manifold $G_{\Psi \setminus \Gamma}(x_{\Gamma, \Sigma})$. It is an indefinite-hermitian symmetric space of $G_{\Psi \setminus \Gamma, 0}$; if $\Gamma \cap \Sigma = \phi$, then its invariant kaehler metric is positive definite if and only if Σ is a (possibly empty) union of sets $(\Psi \setminus \Gamma) \cap \Delta^k$ for which $G_0^k(x_0)$ is of tube type. Its isotropy subalgebra $\mathfrak{G}_{\Psi \setminus \Gamma, 0} \cap \mathcal{O}_{x_{\Gamma, \Sigma}}$ at $x_{\Gamma, \Sigma}$ has complexification $\text{ad}(c_{\Sigma}^2) \{ \mathfrak{G}_{\Psi \setminus \Gamma} \cap \mathcal{O}_{x_{\Gamma, \phi}} \}$.

3. The holomorphic arc components of $G_0(x_{\Gamma, \Sigma})$ are just the $g \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$, $g \in K$. If

(10.10a) $N_{\Gamma, \Sigma, 0} = \{ g \in G_0 : g \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma}) = G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma}) \}$,

(10.10b) $Z_{\Gamma, \Sigma, 0} = \{ g \in N_{\Gamma, \Sigma, 0} : g \text{ acts trivially on } G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma}) \}$,

then

(10.10c) $N_{\Gamma, \Sigma, 0} = G_{\Psi \setminus \Gamma, 0} \cdot Z_{\Gamma, \Sigma, 0}$,

so $G_{\Psi \setminus \Gamma, 0}$ and $N_{\Gamma, \Sigma, 0}$ have the same action on the holomorphic arc component $G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$ of $G_0(x_{\Gamma, \Sigma})$.

4. The parabolic subalgebra $\mathfrak{N}_{[\Gamma, \Sigma]}$ of \mathfrak{G} , complexification of the Lie algebra of $N_{\Gamma, \Sigma, 0}$, is just the parabolic subalgebra that is the sum of the nonnegative eigenspaces of $\text{ad}(\{ \sum_{\Gamma \setminus \Sigma} x_{\gamma, 0} \} - \{ \sum_{\Gamma \cap \Sigma} x_{\gamma, 0} \})$; it is independent of Σ if we require that Γ and Σ be disjoint.

10.11. COROLLARY. *Every G_0 -orbit on X is measurable.*

PROOF OF THEOREM 10.6. If φ is a noncompact root then $\mathfrak{G}[\varphi]$ denotes the simple algebra $\mathfrak{h}_\varphi \mathbf{C} + \mathfrak{G}_\varphi + \mathfrak{G}_{-\varphi}$, and $\mathfrak{G}_0[\varphi]$ denotes its real form $\mathfrak{G}_0 \cap \mathfrak{G}[\varphi]$; $G[\varphi]$ and $G_0[\varphi]$ are the analytic subgroups for $\mathfrak{G}[\varphi]$ and $\mathfrak{G}_0[\varphi]$; and $S[\varphi]$ is the Riemann sphere $G[\varphi](x_0)$. The $G_0[\varphi]$ -orbits on $S[\varphi]$ are:

lower hemisphere $G_0[\varphi] \left(\left\{ \exp \frac{\pi}{4} y_\varphi \right\}^n x_0 \right)$, $n \equiv 0 \pmod{4}$,

equator $G_0[\varphi] \left(\left\{ \exp \frac{\pi}{4} y_\varphi \right\}^n x_0 \right)$, $n \equiv 1 \pmod{2}$,

upper hemisphere $G_0[\varphi] \left(\left\{ \exp \frac{\pi}{4} y_\varphi \right\}^n x_0 \right)$, $n \equiv 2 \pmod{4}$.

Define $\mathfrak{g}[\Psi] = \sum_{\psi} \mathfrak{g}[\psi]$, so the real form $\mathfrak{g}_0[\Psi] = \mathfrak{g}_0 \cap \mathfrak{g}[\Psi]$ is $\sum_{\psi} \mathfrak{g}_0[\psi]$; then the analytic subgroups for $\mathfrak{g}[\Phi]$ and $\mathfrak{g}_0[\Psi]$ are $G[\Psi] = \prod_{\psi} G[\psi]$ and $G_0[\Psi] = \prod_{\psi} G_0[\psi]$, and $S[\Psi] = G[\Psi](x_0)$ is the product $\prod_{\psi} S[\psi]$ of $|\Psi|$ Riemann spheres. Now the $G_0[\Psi]$ -orbits on $S[\Psi]$ are the products $\prod_{\psi} D_{\psi}$ where D_{ψ} is a $G_0[\psi]$ -orbit on $S[\psi]$. Given such an orbit, define

$$\Gamma = \{\psi \in \Psi: D_{\psi} \text{ is the equator of } S[\psi]\}, \text{ and}$$

$$\Sigma = \{\psi \in \Psi: D_{\psi} \text{ is the upper hemisphere of } S[\psi]\}.$$

Then the orbit is $G_0[\Psi](c_{\Gamma}c_{\Sigma}^2x_0) = G_0[\Psi](x_{\Gamma, \Sigma})$, and Γ and Σ are disjoint.

The equator of $S[\psi]$ has descriptions $G_0[\psi](c_{\psi}x_0) = G_0[\psi](c_{\psi}^3x_0)$. If Γ and Σ are arbitrary subsets of Ψ , it follows that $x_{\Gamma, \Sigma} = c_{\Gamma \setminus \Sigma} \cdot c_{\Gamma \cap \Sigma}^2 \cdot c_{\Sigma \setminus \Gamma}^2 x_0 \in G_0[\Psi](x_{\Gamma, \Sigma \setminus \Gamma})$. Thus

$$(10.12a) \quad G_0[\Psi](x_{\Gamma, \Sigma}) = G_0[\Psi](x_{\Gamma, \Sigma \setminus \Gamma}).$$

If we also have subsets Γ' and Σ' of Ψ , it follows that

$$(10.12b) \quad G_0[\Psi](x_{\Gamma, \Sigma}) = G_0[\Psi](x_{\Gamma', \Sigma'}) \text{ if and only if}$$

$$(\Sigma \setminus \Gamma) = (\Sigma' \setminus \Gamma') \text{ and } \Gamma = \Gamma'.$$

It also follows from (10.12a) that $G_0[\Psi](x_{\Gamma', \Sigma'})$ is in the closure of $G_0[\Psi](x_{\Gamma, \Sigma})$ if and only if

- (i) $\Gamma \subset \Gamma'$,
- (ii) $(\Sigma \setminus \Gamma) \subset (\Sigma' \cup \Gamma')$, and
- (iii) $\{\Psi \setminus (\Sigma \cup \Gamma)\} \subset \{\Psi \setminus (\Sigma' \cup \Gamma')\} \cup \Gamma'$.

Those three conditions can be phrased

$$(a) \quad (\Sigma' \setminus \Gamma') \subset (\Sigma \setminus \Gamma) \text{ and } (b) \quad (\Sigma \cup \Gamma) \subset (\Sigma' \cup \Gamma').$$

Thus

$$(10.12c) \quad G_0[\Psi](x_{\Gamma', \Sigma'}) \text{ is in the closure of } G_0[\Psi](x_{\Gamma, \Sigma})$$

$$\text{if and only if } (\Sigma' \setminus \Gamma') \subset (\Sigma \setminus \Gamma) \text{ and } (\Sigma \cup \Gamma) \subset (\Sigma' \cup \Gamma').$$

Finally we observe that

$$(10.12d) \quad G_0[\Psi](x_{\Gamma, \Sigma}) \text{ is open in } S[\Psi] \text{ if and only if } \Gamma = \emptyset,$$

$$(10.12e) \quad G_0[\Psi](x_{\Gamma, \Sigma}) \text{ is closed in } S[\Psi] \text{ if and only if } \Gamma = \Psi.$$

To carry results from $G_0[\Psi]$ -orbits on $S[\Psi]$ to G_0 -orbits on X , we define $A = \exp J\mathfrak{A}_u \subset G_u$. As $J\mathfrak{A}_u$ is a maximal abelian subspace of \mathfrak{M}_u we have $G_u = KAK$. Thus

$$X = G_u(x_0) = KAK(x_0) = KA(x_0) = K \cdot S[\Psi].$$

If $x \in X$ we thus have $x = k(s)$, $s \in S[\Psi]$, so $G_0(x) = G_0(s) = G_0 \cdot G_0[\Psi](s)$. We have disjoint sets Γ and Σ in Ψ with $G_0[\Psi](s) = G_0[\Psi](x_{\Gamma, \Sigma})$. It follows that $G_0(x) = G_0(x_{\Gamma, \Sigma})$. That proves part 1 of Theorem 10.6.

Theorem 3.3 says that X has just one closed G_0 -orbit. In [2, Theorem 3.6], $G_0(x_{\Psi, \emptyset})$ is shown to be the closed G_0 -orbit on the boundary

of X_0 and is identified as the Bergman-Silov boundary of X_0 in X . That proves part 5 of Theorem 10.6.

Let $\Sigma \subset \Psi$. We have $\mathfrak{J} \subset \mathfrak{G}_0 \cap \mathcal{O}_{x_0}$, and (10.4c) with $\Gamma = \phi$ says $\mathfrak{J} = \text{ad}(c_\Sigma^2)\mathfrak{J} \subset \text{ad}(c_\Sigma^2)\mathcal{O}_{x_0} = \mathcal{O}_{c_\Sigma^2 x_0} = \mathcal{O}_{x_\phi, \Sigma}$. Thus $\mathfrak{G}_0 \cap \mathcal{O}_{x_\phi, \Sigma}$ contains a compact Cartan subalgebra \mathfrak{J} of \mathfrak{G}_0 , so Theorem 4.5 says that $G_0(x_\phi, \Sigma)$ is open in X . Conversely, if $G_0(x_\Gamma, \Sigma)$ is open in X , then $G_0(x_\Gamma, \Sigma) \cap S[\Psi]$ is an open $G_0[\Psi]$ -invariant subset of $S[\Psi]$ that contains x_Γ, Σ . As it is open and $G_0[\Psi]$ -invariant, it is a union of sets $G_0[\Psi](x_\phi, B)$, $B \subset \Psi$. Thus we have $B \subset \Psi$ with $x_\Gamma, \Sigma \in G_0[\Psi](x_\phi, B)$. It follows that the open G_0 -orbits on X are just the $G_0(x_\phi, \Sigma)$ with $\Sigma \subset \Psi$.

Let Σ and Σ' be two subsets of Ψ . If $|\Sigma \cap \Delta^k| = |\Sigma' \cap \Delta^k|$ for $1 \leq k \leq p$, then [7, Theorem 2] there is an element $g \in K$ normalizing \mathfrak{J} such that $\text{ad}(g)\Sigma = \Sigma'$, so $G_0(x_\phi, \Sigma') = G_0(c_\Sigma^2 x_0) = G_0(g c_\Sigma^2 g^{-1} x_0) = G_0(c_\Sigma^2 x_0) = G_0(x_\phi, \Sigma)$. Conversely suppose $G_0(x_\phi, \Sigma) = G_0(x_\phi, \Sigma')$. Then, as both orbits are open, Theorem 4.9 says that c_Σ^2 and $c_{\Sigma'}^2$ represent the same double coset in a certain space $W_K \backslash W_G^{\mathfrak{K}_0} / W_{P^r \cap \Gamma, P^r}^{\mathfrak{K}_0}$. There $\mathfrak{K}_0 = \mathfrak{J}$, so $W_G^{\mathfrak{K}_0} = W_G$; and $P^r = \tau P^r = K^c$, so $W_{P^r \cap \Gamma, P^r}^{\mathfrak{K}_0} = W_K$. Thus $c_{\Sigma'}^2 \in W_K \cdot c_\Sigma^2 \cdot W_K$, so we have $w_i \in W_K$ with $\text{ad}(c_{\Sigma'}^2) = w_1 \cdot \text{ad}(c_\Sigma^2) \cdot w_2$. If z denotes the central element of \mathfrak{K} such that $J = \text{ad}(z) | \mathfrak{m}$, and if $\Sigma'' \subset \Psi$, then $\text{ad}(c_{\Sigma''}^2)z - z = \sum_{\psi \in \Sigma''} (i h_\psi)$. Thus

$$\begin{aligned} - \sum_{\psi \in \Sigma'} h_\psi &= i \{ \text{ad}(c_{\Sigma'}^2)z - z \} = i \{ w_1 \cdot \text{ad}(c_\Sigma^2) \cdot w_2 z - z \} \\ &= i \{ w_1 \cdot \text{ad}(c_\Sigma^2)z - z \} = w_1 \cdot i \{ \text{ad}(c_\Sigma^2)z - z \} \\ &= w_1 \left(- \sum_{\psi \in \Sigma} h_\psi \right). \end{aligned}$$

As $\frac{1}{2} \sum_{\psi \in \Sigma'} x_{\psi, 0} = -(i/2) \sum_{\psi \in \Sigma''} x_\psi = \text{ad}(c_{\Sigma''}) \left(- \sum_{\psi \in \Sigma''} h_\psi \right)$, it follows that $\frac{1}{2} \sum_{\psi \in \Sigma'} x_{\psi, 0}$ and $\frac{1}{2} \sum_{\psi \in \Sigma} x_{\psi, 0}$ are conjugate elements of \mathfrak{g} . Thus the corresponding parabolic subalgebras of \mathfrak{g} are conjugate. But [15, Theorem 6.5] those parabolic subalgebras of \mathfrak{g} are the complex normalizers of the boundary components of types $\Psi \setminus \Sigma$ and $\Psi \setminus \Sigma'$ of X_0 in X , so [15, Corollary 6.9] $G_0(x_\Sigma, \phi) = G_0(x_{\Sigma'}, \phi)$, and it follows [15, Lemma 5.1] that $|\Sigma \cap \Delta^k| = |\Sigma' \cap \Delta^k|$ for $1 \leq k \leq p$. In conclusion, open orbits $G_0(x_\phi, \Sigma)$ and $G_0(x_\phi, \Sigma')$ are equal if and only if, for $1 \leq k \leq p$, $|\Sigma \cap \Delta^k| = |\Sigma' \cap \Delta^k|$. In particular, if $r_k = |\Psi \cap \Delta^k|$ then there are precisely $\sum_{1 \leq k \leq p} (r_k + 1) = |\Psi| + p$ open G_0 -orbits on X . That completes the proof of part 4 of Theorem 10.6.

Consider an open orbit $G_0(x_\phi, \Sigma)$. The intersection with $S[\Psi]$ is $G_0[\Psi]$ -invariant and open in $S[\Psi]$, hence a union of sets $G_0[\Psi](x_\phi, B)$. The sets B there are just the subsets of Ψ such that $G_0(x_\phi, \Sigma) = G_0(x_\phi, B)$.

Thus

$$(10.13a) \quad G_0(x_{\phi, z}) \cap S[\Psi] = \cup_{\mathfrak{B}} G_0[\Psi](x_{\phi, B})$$

where

$$(10.13b) \quad \mathfrak{B} = \{ B \subset \Psi : |B \cap \Delta^k| = |\Sigma \cap \Delta^k| \text{ for } 1 \leq k \leq p \}.$$

Now consider an orbit $G_0(x_{\Gamma', z'})$. It is in the closure of $G_0(x_{\phi, z})$ if and only if $x_{\Gamma', z'}$ is in that closure. So (10.12c) says that

$$(10.13c) \quad G_0(x_{\Gamma', z'}) \text{ is in the closure of } G_0(x_{\phi, z}) \text{ if and only if, for some } B \in \mathfrak{B}, (\Sigma' \setminus \Gamma') \subset B \subset (\Sigma' \cup \Gamma').$$

In view of (10.13b), we reformulate (10.13c) as

$$(10.13d) \quad G_0(x_{\Gamma', z'}) \text{ is in the closure of } G_0(x_{\phi, z}) \text{ if and only if, for } 1 \leq k \leq p, |(\Sigma' \setminus \Gamma') \cap \Delta^k| \leq |\Sigma \cap \Delta^k| \leq |(\Sigma' \cup \Gamma') \cap \Delta^k|.$$

Now consider arbitrary orbits $G_0(x_{\Gamma, z})$ and $G_0(x_{\Gamma', z'})$. According to (10.13d), $G_0(x_{\Gamma, z})$ is in the closure of an open orbit $G_0(x_{\phi, z^*})$ if and only if

$$|(\Sigma \setminus \Gamma) \cap \Delta^k| \leq |\Sigma^* \cap \Delta^k| \leq |(\Sigma \cup \Gamma) \cap \Delta^k| \quad \text{for } 1 \leq k \leq p.$$

In particular, $G_0(x_{\Gamma, z})$ is in the closures of the open orbits

$$G_0(x_{\phi, z \setminus \Gamma}) \text{ and } G_0(x_{\phi, z \cup \Gamma}).$$

If $G_0(x_{\Gamma', z'})$ is in the closure of $G_0(x_{\Gamma, z})$, then it is in the closures of $G_0(x_{\phi, z \setminus \Gamma})$ and $G_0(x_{\phi, z \cup \Gamma})$, and (10.13d) says that $|(\Sigma' \setminus \Gamma') \cap \Delta^k| \leq |(\Sigma \setminus \Gamma) \cap \Delta^k|$ and $|(\Sigma' \cup \Gamma') \cap \Delta^k| \leq |(\Sigma \cup \Gamma) \cap \Delta^k|$ for $1 \leq k \leq p$. Conversely, if we have those numerical inequalities, then [7, Theorem 2] there is an element $g \in K$ normalizing \mathfrak{J} such that $\text{ad}(g)$ sends $\Sigma' \setminus \Gamma'$ into $\Sigma \setminus \Gamma$ and sends $\Sigma' \cup \Gamma'$ to a set containing $\Sigma \cup \Gamma$; as $G_0(x_{\Gamma', z'}) = G_0(x_{\text{ad}(g)\Gamma', \text{ad}(g)z'})$ we thus may assume $(\Sigma' \setminus \Gamma') \subset (\Sigma \setminus \Gamma)$ and $(\Sigma' \cup \Gamma') \subset (\Sigma \cup \Gamma)$, so (10.12c) says that $G_0(x_{\Gamma', z'})$ is in the closure of $G_0(x_{\Gamma, z})$. That completes the proof of part 3 of Theorem 10.6.

Orbits $G_0(x_{\Gamma, z})$ and $G_0(x_{\Gamma', z'})$ are equal if and only if each is in the closure of the other. According to part 3 just proved, that can be formulated

$$\begin{aligned} |(\Sigma' \setminus \Gamma') \cap \Delta^k| &= |(\Sigma \setminus \Gamma) \cap \Delta^k| \quad \text{and} \\ |(\Sigma' \cup \Gamma') \cap \Delta^k| &= |(\Sigma \cup \Gamma) \cap \Delta^k|, \end{aligned}$$

which is equivalent to the criterion

$$|(\Sigma' \setminus \Gamma') \cap \Delta^k| = |(\Sigma \setminus \Gamma) \cap \Delta^k| \quad \text{and} \quad |\Gamma' \cap \Delta^k| = |\Gamma \cap \Delta^k|$$

asserted in part 2. In particular, if we enumerate $\Psi \cap \Delta^k = \{\psi_{k,1}, \dots, \psi_{k,r_k}\}$, and the choices of $\Gamma \cap \Delta^k$ and $\Sigma \cap \Delta^k$ leading to distinct G_0 -orbits are

$$\begin{aligned} \Gamma \cap \Delta^k &= \{\psi_{k,1}, \dots, \psi_{k,j}\}, & 0 \leq j \leq r_k, \\ \Sigma \cap \Delta^k &= \{\psi_{k,j+1}, \dots, \psi_{k,l}\}, & j \leq l \leq r_k. \end{aligned}$$

The number of such choices is

$$\begin{aligned} \sum_{j=0}^{r_k} (r_k + 1 - j) &= (r_k + 1)^2 - \sum_{j=0}^{r_k} j = (r_k + 1)^2 - \frac{1}{2}r_k(r_k + 1) \\ &= \frac{1}{2}(r_k + 1)(r_k + 2). \end{aligned}$$

Thus there are precisely $\frac{1}{2} \sum_{k=0}^p (r_k + 1)(r_k + 2)$ G_0 -orbits on X . That proves part 2 of Theorem 10.6, completing the proof of the theorem.

Q.E.D.

PROOF OF THEOREM 10.7. Decompositions (10.1) and (10.2) show that \mathfrak{K}_Γ is an abelian subalgebra of \mathfrak{K}_0 with the same dimension as \mathfrak{J} . If σ is the Cartan involution of \mathfrak{G}_0 with fixed point set \mathfrak{K} , then $\sigma(\mathfrak{K}_\Gamma) = \mathfrak{K}_\Gamma$ by (10.2), so \mathfrak{K}_Γ is a reductive subalgebra of \mathfrak{G}_0 . It follows that \mathfrak{K}_Γ is a Cartan subalgebra of \mathfrak{G}_0 . Note $\mathfrak{K}_\Gamma^C = \text{ad}(c_\Gamma)\mathfrak{J}^C = \text{ad}(c_\Gamma)\text{ad}(c_\Sigma^2)\mathfrak{J}^C \subset \text{ad}(c_\Gamma c_\Sigma^2)\mathfrak{P}_{x_0} = \mathfrak{P}_{c_\Gamma c_\Sigma^2 x_0} = \mathfrak{P}_{x_\Gamma, \Sigma}$; thus \mathfrak{K}_Γ is a Cartan subalgebra of \mathfrak{G}_0 contained in $\mathfrak{P}_{x_\Gamma, \Sigma}$. It is standard [3] that every Cartan subalgebra of \mathfrak{G}_0 is conjugate to one of the \mathfrak{K}_Γ .

Let \mathfrak{C} be maximally compact among the Cartan subalgebras of \mathfrak{G}_0 contained in $\mathfrak{P}_{x_\Gamma, \Sigma}$. Conjugating by an element of $G_0 \cap P_{x_\Gamma, \Sigma}$, we may assume

$$\mathfrak{C} = (\mathfrak{C} \cap \mathfrak{K}) + (\mathfrak{C} \cap \mathfrak{M}_0), \quad (\mathfrak{J}^+ + \mathfrak{J}_\Gamma^-) \subset (\mathfrak{C} \cap \mathfrak{K}), \quad (\mathfrak{C} \cap \mathfrak{M}_0) \subset \mathfrak{A}_\Gamma.$$

If the latter two inclusions are not equalities, then we have $0 \neq v \in \mathfrak{C} \cap \mathfrak{J}_\Gamma^- \setminus \mathfrak{J}_\Psi^-$. Thus $0 \neq v \in \mathfrak{P}_{x_\Gamma, \Sigma} \cap \mathfrak{J}_\Psi^- \setminus \mathfrak{J}_\Gamma^-$. But $\text{ad}(c_\Gamma^{-1})\text{ad}(c_\Sigma^{-2})$ sends $\mathfrak{P}_{x_\Gamma, \Sigma}$ to \mathfrak{P} and sends $\mathfrak{J}_\Psi^- \setminus \mathfrak{J}_\Gamma^-$ to $i\mathfrak{A}_\Gamma$; thus $0 \neq \text{ad}(c_\Gamma^{-1})\text{ad}(c_\Sigma^{-2})v \in \mathfrak{P} \cap i\mathfrak{A}_\Gamma$, contradicting $\mathfrak{P} \cap i\mathfrak{A}_\Gamma = 0$. Thus

$$(\mathfrak{C} \cap \mathfrak{K}) = (\mathfrak{J}^+ + \mathfrak{J}_\Gamma^-) \quad \text{and} \quad (\mathfrak{C} \cap \mathfrak{M}_0) = \mathfrak{A}_\Gamma,$$

proving $\mathfrak{C} = \mathfrak{K}_\Gamma$. Now the first part of Theorem 10.7 is proved.

If \mathfrak{K}_Γ and $\mathfrak{K}_{\Gamma'}$ are conjugate in \mathfrak{G}_0 , then for $1 \leq k \leq p$ their split parts $\mathfrak{A}_\Gamma \cap \mathfrak{G}^k$ and $\mathfrak{A}_{\Gamma'} \cap \mathfrak{G}^k$ along the simple ideal $\mathfrak{G}_0 \cap \mathfrak{G}^k$ of \mathfrak{G}_0 must have the same dimension; those dimensions are $|\Gamma \cap \Delta^k|$ and $|\Gamma' \cap \Delta^k|$, which thus must be equal. Conversely, if $|\Gamma \cap \Delta^k| = |\Gamma' \cap \Delta^k|$ for $1 \leq k \leq p$, then [7, Theorem 2], we have $g \in K$ normalizing \mathfrak{J} with $\text{ad}(g)\mathfrak{K}_{\Gamma'} = \mathfrak{K}_\Gamma$. That proves part 2 of Theorem 10.7. Parts 4 and 5 are immediate, for \mathfrak{K}_Γ is maximally compact if and only if it is conjugate to $\mathfrak{J} = \mathfrak{K}_\phi$, maximally split if and only if it is conjugate to $\mathfrak{J}^+ + \mathfrak{A}_0 = \mathfrak{K}_\Psi$.

If \mathfrak{K}_Γ centralizes a G_0 -conjugate $\text{ad}(g)\mathfrak{J}_\Gamma^-$, then $\text{ad}(g)\mathfrak{J}_\Gamma^- \subset \mathfrak{K}_\Gamma$. As \mathfrak{K}_Γ is commutative, now $\text{ad}(g)\mathfrak{J}_\Gamma^- \subset \mathfrak{K} \cap \mathfrak{K}_\Gamma = \mathfrak{J}^+ + \mathfrak{J}_\Gamma^- \subset \mathfrak{J}$, so we can assume that $g \in K$ and $\text{ad}(g)\mathfrak{J} = \mathfrak{J}$; thus $\mathfrak{A}_\Gamma + J\mathfrak{A}_\Gamma + [\mathfrak{A}_\Gamma, J\mathfrak{A}_\Gamma] = \mathfrak{G}_0[\Gamma]$ centralizes $\text{ad}(g)\mathfrak{J}_\Gamma^-$, so $\mathfrak{G}_0[\Gamma] \subset \text{ad}(g)(\mathfrak{G}_{\Gamma', 0})$. The former has Γ , and the latter has $\text{ad}(g)\Gamma'$, as maximal set of strongly orthogonal non-

compact roots. Thus, for $1 \leq k \leq p$, $|\Gamma \cap \Delta^k| \leq |\text{ad}(g)\Gamma' \cap \Delta^k| = |\Gamma' \cap \Delta^k|$. Conversely, if $|\Gamma \cap \Delta^k| \leq |\Gamma' \cap \Delta^k|$ for $1 \leq k \leq p$, then we conjugate Γ' by an element of W so that $\Gamma \subset \Gamma'$, and then \mathfrak{K}_Γ centralizes \mathfrak{J}_Γ . That proves part 3, completing the proof of Theorem 10.7. Q.E.D.

PROOF OF THEOREM 10.9. Our assertions all are known for G_0 -orbits on the boundary of $X_0 = G_0(x_0)$ in X : in Wolf-Korányi [15] it is shown that $G_{\Psi \setminus \Gamma}(x_0)$ is a compact totally geodesic hermitian symmetric subspace of X (Lemma 4.2); that $G_{\Psi \setminus \Gamma, 0}(x_0) \subset G_{\Psi \setminus \Gamma}(x_0)$ is the Borel embedding of the noncompact dual (Lemma 4.2); that $\Psi \setminus \Gamma$ is a maximal set of strongly orthogonal roots of $\mathfrak{G}_{\Psi \setminus \Gamma, 0}$ relative to $\mathfrak{J} \cap \mathfrak{G}_{\Psi \setminus \Gamma, 0}$ (Lemma 4.2); that the holomorphic arc components of the topological boundary ∂X_0 of X_0 in X are just the sets $g \cdot c_\Gamma G_{\Psi \setminus \Gamma, 0}(x_0)$ with $g \in K$ and $\phi \neq \Gamma \subset \Psi$ (Theorem 4.8); and that the set $B_{\Psi \setminus \Gamma}$ of all elements of G_0 that preserve $c_\Gamma G_{\Psi \setminus \Gamma, 0}(x_0)$ is the parabolic subgroup of G_0 whose Lie algebra $\mathfrak{B}_{\Psi \setminus \Gamma}$ is the sum of the nonnegative eigenspaces of $\text{ad}(\sum_\Gamma x_{\gamma, 0})$ on \mathfrak{G}_0 and satisfies $\mathfrak{B}_{\Psi \setminus \Gamma} = \mathfrak{G}_{\Psi \setminus \Gamma, 0} + \mathfrak{Z}_{\Psi \setminus \Gamma}$ where $\mathfrak{Z}_{\Psi \setminus \Gamma}$ is the ideal that is the Lie algebra of the kernel of the action of $B_{\Psi \setminus \Gamma}$ on $c_\Gamma G_{\Psi \setminus \Gamma, 0}(x_0)$ (Theorems 6.5 and 6.8). Now we make the key observation that every holomorphic arc component of ∂X_0 is contained in some G_0 -orbit on ∂X_0 , and thus is a holomorphic arc component of that G_0 -orbit. Thus the case $\Sigma = \phi$ of Theorem 10.9, i.e. the case of G_0 -orbits in the closure of X_0 , is contained in the results of Wolf-Korányi [15].

Now we go to an arbitrary orbit $G_0(x_{\Gamma, \Sigma})$. To minimize subscripts on subscripts, we adopt the notation

$$\mathcal{O}_{\Gamma, \Sigma} = \mathcal{O}_{x_{\Gamma, \Sigma}} \quad \text{and} \quad \mathfrak{K}_{\Gamma, \Sigma} = \mathfrak{K}_{[x_{\Gamma, \Sigma}]}$$

Recall the Cartan subalgebra $\mathfrak{K}_\Gamma \subset \mathcal{O}_{\Gamma, \Sigma}$ of \mathfrak{G}_0 . All roots in this paragrah will be \mathfrak{K}_Γ^c -roots of \mathfrak{g} . Now

$$\begin{aligned} \Phi_{\Gamma, \Sigma} &\text{ denotes the root system of } \mathcal{O}_{\Gamma, \Sigma}, \\ \Lambda_{\Gamma, \Sigma} &\text{ denotes the root system of } \mathfrak{K}_{\Gamma, \Sigma}. \end{aligned}$$

From (10.3) and (10.4) we have

$$\text{ad}(c_\Sigma^2)\mathfrak{K}_\Gamma = \mathfrak{K}_\Gamma \quad \text{and} \quad \tau \cdot \text{ad}(c_\Sigma^2)|_{\mathfrak{K}_\Gamma} = \text{ad}(c_\Sigma^2) \cdot \tau|_{\mathfrak{K}_\Gamma}$$

This and $\text{ad}(c_\Sigma^2)\mathcal{O}_{\Gamma, \phi} = \mathcal{O}_{\Gamma, \Sigma}$ imply

$$(10.14a) \quad \text{ad}(c_\Sigma^2)\Phi_{\Gamma, \phi} = \Phi_{\Gamma, \Sigma} \quad \text{and} \quad \text{ad}(c_\Sigma^2)\tau\Phi_{\Gamma, \phi} = \tau\Phi_{\Gamma, \Sigma};$$

in particular

$$(10.14b) \quad \text{ad}(c_\Sigma^2)\{\Phi_{\Gamma, \phi} \cap \tau\Phi_{\Gamma, \phi}\} = \Phi_{\Gamma, \Sigma} \cap \tau\Phi_{\Gamma, \Sigma},$$

$$(10.14c) \quad \text{ad}(c_{\Sigma}^2)\{\Phi_{\Gamma,\phi} \cup \tau\Phi_{\Gamma,\phi}\} = \Phi_{\Gamma,\Sigma} \cup \tau\Phi_{\Gamma,\Sigma}.$$

Recall Theorem 8.5, let $\delta_{\Gamma,\Sigma}$ denote the linear form (8.6a) on $\mathfrak{H}_{\Gamma}^{\mathbb{C}}$, and let $\mathcal{Q}_{\Gamma,\Sigma}$ denote the parabolic subalgebra (8.6c) for $x_{\Gamma,\Sigma}$. Then (10.14a) gives us

$$\text{ad}(c_{\Sigma}^2)\delta_{\Gamma,\phi} = \text{ad}(c_{\Sigma}^2) \sum_{\Phi_{\Gamma,\phi}^u \cap \tau\Phi_{\Gamma,\phi}^u} \varphi = \sum_{\Phi_{\Gamma,\Sigma}^u \cap \tau\Phi_{\Gamma,\Sigma}^u} \varphi = \delta_{\Gamma,\Sigma}$$

so $\text{ad}(c_{\Sigma}^2)\mathcal{Q}_{\Gamma,\phi} = \mathcal{Q}_{\Gamma,\Sigma}$. Recall Proposition 8.7 and let $\mathfrak{N}_{\Gamma,\Sigma}$ denote the linear space (8.8b) for $x_{\Gamma,\Sigma}$. In view of (10.14c) it follows that $\text{ad}(c_{\Sigma}^2)\mathfrak{N}_{\Gamma,\phi} = \mathfrak{N}_{\Gamma,\Sigma}$. As $G_0(x_{\Gamma,\phi})$ is partially complex, Theorem 8.9 says $\mathfrak{N}_{\Gamma,\phi} = \mathfrak{N}_{\Gamma,\phi}$, so $\mathfrak{N}_{\Gamma,\phi}$ is an algebra, and thus $\mathfrak{N}_{\Gamma,\Sigma} = \text{ad}(c_{\Sigma}^2)\mathfrak{N}_{\Gamma,\phi}$ is an algebra; now $\mathfrak{N}_{\Gamma,\Sigma} = \mathfrak{N}_{\Gamma,\Sigma}$ by Theorem 8.9. Thus

$$(10.15a) \quad \mathfrak{N}_{\Gamma,\Sigma} = \text{ad}(c_{\Sigma}^2)\mathfrak{N}_{\Gamma,\phi} \quad \text{and} \quad N_{\Gamma,\Sigma} = \text{ad}(c_{\Sigma}^2)N_{\Gamma,\phi}$$

where the latter are analytic subgroups of G . Now let $Z_{\Gamma,\Sigma}$ be the kernel of the action of $N_{\Gamma,\Sigma}$ on $N_{\Gamma,\Sigma}(x_{\Gamma,\Sigma})$, $Z_{\Gamma,\Sigma}$ its Lie algebra. Then (10.15a) gives us

$$(10.15b) \quad Z_{\Gamma,\Sigma} = \text{ad}(c_{\Sigma}^2)Z_{\Gamma,\phi} \quad \text{and} \quad Z_{\Gamma,\Sigma} = \text{ad}(c_{\Sigma}^2)Z_{\Gamma,\phi}.$$

As (10.4a) says $\text{ad}(c_{\Sigma}^2)\mathfrak{J}_{\Psi\backslash\Gamma} = \mathfrak{J}_{\Psi\backslash\Gamma}$, (10.8a) yields

$$(10.15c) \quad \text{ad}(c_{\Sigma}^2)\mathfrak{G}_{\Psi\backslash\Gamma} = \mathfrak{G}_{\Psi\backslash\Gamma}.$$

We already know $\mathfrak{N}_{\Gamma,\phi} = \mathfrak{G}_{\Psi\backslash\Gamma} + Z_{\Gamma,\phi}$; thus (10.15a, b, c) combine for

$$(10.15d) \quad \mathfrak{N}_{\Gamma,\Sigma} = \mathfrak{G}_{\Psi\backslash\Gamma} + Z_{\Gamma,\Sigma} \quad \text{and} \quad N_{\Gamma,\Sigma} = G_{\Psi\backslash\Gamma} \cdot Z_{\Gamma,\Sigma}.$$

Now (10.10c) follows. That proves part 3 of Theorem 10.9. As $\mathfrak{N}_{\Gamma,\phi}$ is the sum of the nonnegative eigenspaces of $\text{ad}\{\sum_{\Gamma} x_{\gamma,0}\}$ on \mathfrak{G} , (10.15a) says that $\mathfrak{N}_{\Gamma,\Sigma}$ is the sum of the nonnegative eigenspaces of

$$\text{ad}\left\{\text{ad}(c_{\Sigma}^2) \sum_{\Gamma} x_{\gamma,0}\right\} = \text{ad}\left(\left\{\sum_{\Gamma\backslash\Sigma} x_{\gamma,0}\right\} - \left\{\sum_{\Gamma\cap\Sigma} x_{\gamma,0}\right\}\right).$$

That proves part 4 of Theorem 10.9.

Note that c_{Γ} centralizes $G_{\Psi\backslash\Gamma}$. Using (10.15c), we have

$$G_{\Psi\backslash\Gamma}(x_{\Gamma,\Sigma}) = c_{\Sigma}^2 G_{\Psi\backslash\Gamma}(x_{\Gamma,\phi}) = c_{\Gamma} c_{\Sigma}^2 G_{\Psi\backslash\Gamma}(x_0).$$

We already know that $G_{\Psi\backslash\Gamma}(x_{\Gamma,\phi})$ is a compact totally geodesic hermitian symmetric subspace of X , rank $|\Psi\backslash\Gamma|$ with $\Psi\backslash\Gamma$ as maximal set of strongly orthogonal noncompact roots of $\mathfrak{G}_{\Psi\backslash\Gamma,0}$ relative to $\mathfrak{J} \cap \mathfrak{G}_{\Psi\backslash\Gamma,0}$. Now the same statements follow for $G_{\Psi\backslash\Gamma}(x_{\Gamma,\Sigma})$. That proves part 1 of Theorem 10.9.

$G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$ is open in $G_{\Psi \setminus \Gamma}(x_{\Gamma, \Sigma})$ by (10.15d) and the fact that $G_0(x_{\Gamma, \Sigma})$ is partially complex. $G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$ is symmetric with invariant kaehler metric by Corollary 9.25. If $\Gamma \cap \Sigma = \phi$ and Σ is a (possibly empty) union of sets $(\Psi \setminus \Gamma) \cap \Delta^k$ for which $G_0^k(x_0)$ is of tube type, then $G_{\Psi \setminus \Gamma, 0}$ has the same (thus compact) isotropy subgroups at $x_{\Gamma, \phi}$ and $x_{\Gamma, \Sigma}$, so the metric is positive definite. If $\Gamma \cap \Psi = \phi$ and the metric is positive definite, then $G_{\Psi \setminus \Gamma, 0}$ has the same isotropy subgroups at $x_{\Gamma, \phi}$ and $x_{\Gamma, \Sigma}$, so [15, Lemma 5.6] the (-1) -eigenspace of $\text{ad}(c_{\Sigma}^k)$ on $\mathfrak{g}_{\Psi \setminus \Gamma}$ is zero, and the last statement of [15, Theorem 5.7] says that Σ is a union of sets $(\Psi \setminus \Gamma) \cap \Delta^k$ for which $G_{\Psi \setminus \Gamma, 0}^k(x_0)$ is of tube type. However, [15, Corollary 4.11] says that the latter is equivalent to $G_0^k(x_0)$ being of tube type. That proves part 2, completing the proof of Theorem 10.9. Q.E.D.

11. Hermitian symmetric spaces: compact subvarieties and Siegel domain realizations. Retain the notation and terminology of §10. We will apply our explicit analysis of the G_0 -orbit structure of X to obtain certain “generalized Siegel domain” realizations of an arbitrary orbit $G_0(x_{\Gamma, \Sigma}) \subset X$. Those realizations are composed of several decompositions of $G_0(x_{\Gamma, \Sigma})$ as follows.

1. $\alpha: G_0(x_{\Gamma, \Sigma}) \rightarrow G_0/N_{\Gamma, \Sigma, 0}$ is the K -equivariant fibration whose fibres are the holomorphic arc components $k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma}) = kc_{\Gamma} G_{\Psi \setminus \Gamma, 0}(x_{\phi, \Sigma})$, $k \in K$, of $G_0(x_{\Gamma, \Sigma})$, and whose base is the space $G_0/N_{\Gamma, \Sigma, 0} = K/(K \cap N_{\Gamma, \Sigma, 0})$ of holomorphic arc components of $G_0(x_{\Gamma, \Sigma})$.

2. $\beta: G_0(x_{\Gamma, \Sigma}) \rightarrow K(x_{\Gamma, \Sigma})$ is a K -equivariant fibration whose restriction β_k to a holomorphic arc component $k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$ is a holomorphic fibration of that arc component over its maximal compact subvariety $k \cdot (K \cap G_{\Psi \setminus \Gamma, 0})(x_{\Gamma, \Sigma})$. Assume Γ and Σ disjoint. Then the fibre of β (and also of β_k) over $kk'(x_{\Gamma, \Sigma})$, $k' \in K \cap G_{\Psi \setminus \Gamma, 0}$, is a certain hermitian symmetric space

$$\begin{aligned} kk' \cdot G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_{\Gamma, \Sigma}) &= kk' c_{\Gamma} c_{\Sigma}^2 \cdot G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_0) \\ &= kk' c_{\Gamma} c_{\Sigma}^2 \{ G_{\Psi \setminus (\Gamma \cup \Sigma), 0}(x_0) \times G_{\Sigma, 0}^{\Sigma}(x_0) \} \end{aligned}$$

of noncompact type and rank $|\Psi \setminus \Gamma|$ where $G_{\Sigma, 0}^{\Sigma}(x_0)$ denotes the tube domain part (c.f. [2]) of the domain $G_{\Sigma, 0}(x_0)$. α factors through β , i.e. $\alpha \cdot \beta^{-1}: K(x_{\Gamma, \Sigma}) \rightarrow G_0/N_{\Gamma, \Sigma, 0}$ is a well-defined K -equivariant fibration.

3. $\gamma^{A, B}: G_0(x_{\Gamma, \Sigma}) \rightarrow W \cdot G_{E', 0}^{B'}$ where W is a certain subset of K and we have disjoint decompositions

$$\Sigma = A \cup A', \quad \Psi \setminus (\Gamma \cup \Sigma) = B \cup B', \quad E = A \cup B, \quad E' = A' \cup B'.$$

Given $k \in W$, the restriction $\gamma_k^{A,B}$ of $\gamma^{A,B}$ to $(\gamma^{A,B})^{-1}(k \cdot G_{E',0}^{B'}(x_{\Gamma,Z}))$ is a holomorphic fibration $\gamma_k^{A,B}: k \cdot G_{\Psi \setminus \Gamma,0}^Z(x_{\Gamma,Z}) \rightarrow k \cdot G_{E',0}^{B'}(x_{\Gamma,Z})$ equivalent to a certain projection that is implicit (but made explicit in the appendix to this §11) in the intrinsic realization (c.f. [15] or the appendix to this §11) of $G_{\Psi \setminus \Gamma,0}^Z(x_0)$ as a Siegel domain of type III over its boundary component $G_{E',0}^{B'}(x_{E,\phi})$. $\gamma^{A,B}$ is not necessarily continuous, but it is real analytic almost everywhere, and β factors through $\gamma^{A,B}$.

The fibration $\alpha: G_0(x_{\Gamma,Z}) \rightarrow G_0/N_{\Gamma,Z,0}$ is the subject of Theorem 8.15 with details supplied by Theorem 10.9. The fibration $\beta: G_0(x_{\Gamma,Z}) \rightarrow K(x_{\Gamma,Z})$ is described in Theorem 11.8 below, while $\gamma^{A,B}$ is described by Proposition 11.20 and Theorem 11.23. Proposition 11.20 is a transcription of Proposition 11.3A (A for appendix) which extends the Wolf-Korányi description [15] of the realizations of hermitian symmetric spaces as Siegel domains of type III.

Let Γ and Σ be disjoint subsets of Ψ . We decompose the real form $\mathfrak{G}_{\Psi \setminus \Gamma,0} = \mathfrak{G}_0 \cap \mathfrak{G}_{\Psi \setminus \Gamma}$ of $\mathfrak{G}_{\Psi \setminus \Gamma}$ by

$$(11.1a) \quad \mathfrak{G}_{\Psi \setminus \Gamma,0} = \mathfrak{K}_{\Psi \setminus \Gamma} + \mathfrak{M}_{\Psi \setminus \Gamma,0}, \quad \mathfrak{K}_{\Psi \setminus \Gamma} = \mathfrak{K} \cap \mathfrak{G}_{\Psi \setminus \Gamma} \quad \text{and} \\ \mathfrak{M}_{\Psi \setminus \Gamma,0} = \mathfrak{M}_0 \cap \mathfrak{G}_{\Psi \setminus \Gamma}.$$

If $\psi \in \Psi$ then $\text{ad}(c_\psi)$ has order 4 on the odd dimensional irreducible representation spaces of $\text{ad}_{\mathfrak{g}}(\mathfrak{g}[\psi])$, order 8 on the even dimensional ones; so $\text{ad}(c_\psi^4)$ has square 1. Now

$$\text{ad}(c_\psi^4) = \text{ad}(c_\psi^4)^{-1} = \tau^{-1} \cdot \text{ad}(c_\psi^4) \cdot \tau$$

shows that $\text{ad}(c_\psi^4)$ preserves \mathfrak{G}_0 , and

$$\text{ad}(c_\psi^4) = \text{ad}(c_\psi^4)^{-1} = \sigma^{-1} \cdot \text{ad}(c_\psi^4) \cdot \sigma$$

shows further that $\text{ad}(c_\psi^4)$ preserves \mathfrak{K} and \mathfrak{M}_0 . Define

$$(11.1b) \quad \mathfrak{G}_{\Psi \setminus \Gamma}^\Sigma = \{v \in \mathfrak{G}_{\Psi \setminus \Gamma}: \text{ad}(c_\Sigma^4)v = v\} \quad \text{and} \quad \mathfrak{G}_{\Psi \setminus \Gamma,0}^\Sigma = \mathfrak{G}_0 \cap \mathfrak{G}_{\Psi \setminus \Gamma}^\Sigma.$$

Now $\mathfrak{G}_{\Psi \setminus \Gamma,0}^\Sigma$ is a real form of $\mathfrak{G}_{\Psi \setminus \Gamma}^\Sigma$, and in fact

$$(11.1c) \quad \mathfrak{G}_{\Psi \setminus \Gamma,0}^\Sigma = \mathfrak{K}_{\Psi \setminus \Gamma}^\Sigma + \mathfrak{M}_{\Psi \setminus \Gamma,0}^\Sigma, \quad \mathfrak{K}_{\Psi \setminus \Gamma}^\Sigma = \mathfrak{K} \cap \mathfrak{G}_{\Psi \setminus \Gamma}^\Sigma \quad \text{and} \\ \mathfrak{M}_{\Psi \setminus \Gamma,0}^\Sigma = \mathfrak{M}_0 \cap \mathfrak{G}_{\Psi \setminus \Gamma}^\Sigma.$$

Finally, we define

$$(11.1d) \quad \mathfrak{Q}_{\Psi \setminus \Gamma}^\Sigma \text{ is the } (-1)\text{-eigenspace of } \text{ad}(c_\Sigma^4) \text{ on } \mathfrak{K}_{\Psi \setminus \Gamma},$$

$$(11.1e) \quad \mathfrak{R}_{\Psi \setminus \Gamma,0}^\Sigma \text{ is the } (-1)\text{-eigenspace of } \text{ad}(c_\Sigma^4) \text{ on } \mathfrak{M}_{\Psi \setminus \Gamma,0}^\Sigma.$$

That gives us symmetric decompositions under $\text{ad}(c_2^4)$:

$$(11.1f) \quad \mathcal{K}_{\Psi \setminus \Gamma} = \mathcal{K}_{\Psi \setminus \Gamma}^{\Sigma} + \mathcal{Q}_{\Psi \setminus \Gamma}^{\Sigma} \quad \text{and} \quad \mathcal{G}_{\Psi \setminus \Gamma, 0} = \mathcal{G}_{\Psi \setminus \Gamma, 0}^{\Sigma} + (\mathcal{Q}_{\Psi \setminus \Gamma}^{\Sigma} + \mathcal{R}_{\Psi \setminus \Gamma, 0}^{\Sigma}).$$

Roman letters denote corresponding analytic groups.

We collect some information in a sequence of lemmas.

11.2. LEMMA. $K_{\Psi \setminus \Gamma} = K \cap G_{\Psi \setminus \Gamma, 0}$, maximal compact subgroup of $G_{\Psi \setminus \Gamma, 0}$.

11.3. LEMMA. $K_{\Psi \setminus \Gamma}^{\Sigma} = K \cap G_{\Psi \setminus \Gamma, 0}^{\Sigma}$, maximal compact subgroup of $G_{\Psi \setminus \Gamma, 0}^{\Sigma}$. Both $K_{\Psi \setminus \Gamma}^{\Sigma}$ and $G_{\Psi \setminus \Gamma, 0}^{\Sigma}$ are normalized by $c_{\Gamma} c_{\Sigma}^2$, and $K_{\Psi \setminus \Gamma}^{\Sigma}$ is the isotropy subgroup of $G_{\Gamma \setminus \Psi, 0}^{\Sigma}$ both at x_0 and at $x_{\Gamma, \Sigma}$. In particular

$$G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_{\Gamma, \Sigma}) = c_{\Gamma} c_{\Sigma}^2 G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_0) \cong G_{\Psi \setminus \Gamma, 0}^{\Sigma} / K_{\Psi \setminus \Gamma}^{\Sigma},$$

and that orbit is a totally geodesic hermitian symmetric subspace of noncompact type and rank $|\Psi \setminus \Gamma|$ in $c_{\Gamma} c_{\Sigma}^2 X_0$.

11.4. LEMMA. $K_{\Psi \setminus \Gamma}^{\Sigma}$ is the isotropy subgroup of $K_{\Psi \setminus \Gamma}$ at $x_{\Gamma, \Sigma}$, and the orbit $K_{\Psi \setminus \Gamma}(x_{\Gamma, \Sigma}) \cong K_{\Psi \setminus \Gamma} / K_{\Psi \setminus \Gamma}^{\Sigma}$ is a totally geodesic hermitian symmetric subspace of compact type in X .

11.5. LEMMA. The isotropy subgroup of $G_{\Psi \setminus \Gamma, 0}$ at $x_{\Gamma, \Sigma}$ is the analytic subgroup with Lie algebra $\mathcal{K}_{\Psi \setminus \Gamma}^{\Sigma} + \mathcal{R}_{\Psi \setminus \Gamma, 0}^{\Sigma} = \text{ad}(c_{\Gamma} c_{\Sigma}^2) \mathcal{K}_{\Psi \setminus \Gamma}^{\Sigma} \cap \mathcal{G}_{\Psi \setminus \Gamma, 0}$.

11.6. LEMMA. Define $f: K_{\Psi \setminus \Gamma} \times \mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\Sigma} \times \mathcal{R}_{\Psi \setminus \Gamma, 0}^{\Sigma} \rightarrow G_{\Psi \setminus \Gamma, 0}$ by $f(k, v_1, v_2) = k \cdot \exp v_1 \cdot \exp v_2$. Then f is a diffeomorphism.

PROOF OF LEMMAS. $G_{\Psi \setminus \Gamma, 0}(x_0) \cong G_{\Psi \setminus \Gamma, 0} / (K \cap G_{\Psi \setminus \Gamma, 0})$ is a hermitian symmetric subspace of noncompact type and rank $|\Psi \setminus \Gamma|$ in X_0 . Thus $K \cap G_{\Psi \setminus \Gamma, 0}$ is connected and maximal compact in $G_{\Psi \setminus \Gamma, 0}$. As $K \cap G_{\Psi \setminus \Gamma, 0}$ has Lie algebra $\mathcal{K}_{\Psi \setminus \Gamma}$, now $K \cap G_{\Psi \setminus \Gamma, 0} = K_{\Psi \setminus \Gamma}$ and Lemma 11.2 is proved.

Note $\mathcal{G}_{\Psi \setminus \Gamma, 0} \subset \mathcal{G}_{\Psi \setminus \Gamma, 0}^{\Sigma} \subset \mathcal{G}_{\Psi \setminus \Gamma}$ and $\text{ad}(z) \mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\Sigma} = \mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\Sigma}$. It follows that $G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_0)$ is a totally geodesic hermitian symmetric submanifold of noncompact type and rank $|\Psi \setminus \Gamma|$ in X_0 . As above, now $K_{\Psi \setminus \Gamma}^{\Sigma} = K \cap G_{\Psi \setminus \Gamma, 0}^{\Sigma}$ isotropy subgroup at x_0 and maximal compact subgroup of $G_{\Psi \setminus \Gamma, 0}^{\Sigma}$.

As c_{Γ} centralizes $G_{\Psi \setminus \Gamma}$ it centralizes the subgroups $K_{\Psi \setminus \Gamma}^{\Sigma}$ and $G_{\Psi \setminus \Gamma, 0}^{\Sigma}$. As $\text{ad}(c_{\Sigma}^2)$ preserves $\mathcal{G}_{\Psi \setminus \Gamma}$ and commutes with $\text{ad}(c_2^4)$, it preserves $\mathcal{G}_{\Psi \setminus \Gamma}^{\Sigma}$. Now $\text{ad}(c_2^2)$ preserves $\mathcal{G}_{\Psi \setminus \Gamma, 0}^{\Sigma}$ because

$$\text{ad}(c_2^2) = \text{ad}(c_2^2)^{-1} = \tau^{-1} \cdot \text{ad}(c_2^2) \cdot \tau \quad \text{on } \mathcal{G}_{\Psi \setminus \Gamma, 0}^{\Sigma}.$$

Furthermore

$$\text{ad}(c_{\Sigma}^2) = \text{ad}(c_{\Sigma}^2)^{-1} = \sigma^{-1} \cdot \text{ad}(c_{\Sigma}^2) \cdot \sigma \quad \text{on } \mathcal{G}_{\Psi \setminus \Gamma, 0}^{\Sigma},$$

so $\text{ad}(c_\Sigma^2)$ preserves both $\mathcal{K}_{\Psi \setminus \Gamma}^{\Sigma}$ and $\mathfrak{M}_{\Psi \setminus \Gamma}^{\Sigma}$. Now $G_{\Psi \setminus \Gamma, 0}^{\Sigma}$ and $K_{\Psi \setminus \Gamma}^{\Sigma}$ are centralized by $\text{ad}(c_\Gamma)$ and normalized by $\text{ad}(c_\Sigma^2)$, so they are normalized by $\text{ad}(c_\Gamma c_\Sigma^2)$. As $K_{\Psi \setminus \Gamma}^{\Sigma}$ is the isotropy subgroup of $G_{\Psi \setminus \Gamma, 0}^{\Sigma}$ at x_0 , it thus is the isotropy subgroup at $c_\Gamma c_\Sigma^2 x_0 = x_{\Gamma, \Sigma}$. Lemma 11.3 is proved.

The symmetry of X at $x_{\Gamma, \Sigma}$ is $\text{ad}(c_\Gamma c_\Sigma^2) \cdot \sigma \cdot \text{ad}(c_\Gamma c_\Sigma^2)^{-1} = \text{ad}(c_\Gamma c_\Sigma^2)^2 \cdot \sigma = \text{ad}(c_\Gamma^2) \cdot \text{ad}(c_\Sigma^4) \cdot \sigma$. It acts on $G_{\Psi \setminus \Gamma, 0}$ as $\text{ad}(c_\Sigma^4) \cdot \sigma$. Thus it normalizes $K_{\Psi \setminus \Gamma}$, so $K_{\Psi \setminus \Gamma}(x_{\Gamma, \Sigma})$ is a compact totally geodesic symmetric subspace of X . It is a complex submanifold of X by Lemma 5.1, so it is hermitian symmetric. The isotropy subgroup of $K_{\Psi \setminus \Gamma}$ at $x_{\Gamma, \Sigma}$ contains $K_{\Psi \setminus \Gamma}^{\Sigma}$, has Lie algebra $\mathfrak{K}_{\Psi \setminus \Gamma}^{\Sigma}$ that contains the center of $\mathfrak{K}_{\Psi \setminus \Gamma}$; thus $K_{\Psi \setminus \Gamma}^{\Sigma}(x_0)$ is of compact type and necessarily simply connected; now $K_{\Psi \setminus \Gamma}^{\Sigma}$ is the isotropy subgroup of $K_{\Psi \setminus \Gamma}$ at $x_{\Gamma, \Sigma}$. Lemma 11.4 is proved.

The isotropy subalgebra of $\mathfrak{G}_{\Psi \setminus \Gamma, 0}$ at $c_\Gamma x_0$ is $\mathfrak{K}_{\Psi \setminus \Gamma}$, so the isotropy subalgebra at $c_\Sigma^2(c_\Gamma x_0) = x_{\Gamma, \Sigma}$ is $\text{ad}(c_\Sigma^2)\mathfrak{K}_{\Psi \setminus \Gamma} \cap \mathfrak{G}_{\Psi \setminus \Gamma, 0}$. Now $\text{ad}(c_\Sigma^2)$ preserves $\mathfrak{K}_{\Psi \setminus \Gamma}^{\Sigma}$, as seen above, and sends $\mathcal{Q}_{\Psi \setminus \Gamma}^{\Sigma}$ to $i\mathcal{R}_{\Psi \setminus \Gamma}^{\Sigma}$ by [15, Lemma 5.6]. Thus $\mathfrak{G}_{\Psi \setminus \Gamma, 0}$ has isotropy subalgebra $\mathfrak{K}_{\Psi \setminus \Gamma}^{\Sigma} + \mathcal{R}_{\Psi \setminus \Gamma, 0}^{\Sigma}$ at $x_{\Gamma, \Sigma}$. The isotropy subgroup of $G_{\Psi \setminus \Gamma, 0}$ at $x_{\Gamma, \Sigma}$ is connected by Theorem 5.4, so it must be the analytic subgroup for $\mathfrak{K}_{\Psi \setminus \Gamma}^{\Sigma} + \mathcal{R}_{\Psi \setminus \Gamma, 0}^{\Sigma}$. Lemma 11.5 is proved.

Define $\varphi: \mathfrak{M}_{\Psi \setminus \Gamma, 0} \rightarrow G_{\Psi \setminus \Gamma, 0}(x_0)$ by $\varphi(v) = \exp(-v_2) \cdot \exp(-v_1) \cdot x_0$ where $v = v_1 + v_2$, $v_1 \in \mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\Sigma}$ and $v_2 \in \mathcal{R}_{\Psi \setminus \Gamma, 0}^{\Sigma}$. We view φ as a differentiable map from the euclidean space $\mathfrak{M}_{\Psi \setminus \Gamma, 0}$ to the symmetric space $G_{\Psi \setminus \Gamma, 0}(x_0)$ of noncompact type. We will prove $\|\varphi_* \xi\| \geq \|\xi\|$ for every tangent vector ξ to $\mathfrak{M}_{\Psi \setminus \Gamma, 0}$. Then [13, Theorem 1] φ is surjective and covering, hence a diffeomorphism because $G_{\Psi \setminus \Gamma, 0}(x_0)$ is simply connected, and it follows that from $f(k, v_1, v_2)^{-1}x_0 = \varphi(v_1 + v_2)$ that f defines a diffeomorphism of $K \times \mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\Sigma} \times \mathcal{R}_{\Psi \setminus \Gamma, 0}^{\Sigma}$ onto $G_{\Psi \setminus \Gamma, 0}$.

Let $v_1 \in \mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\Sigma}$ be fixed, $v_2 \in \mathcal{R}_{\Psi \setminus \Gamma, 0}^{\Sigma}$ variable. Then $\varphi(v_1 + v_2) = \exp(-v_2) \exp(-v_1)x_0 = \exp(-v_1) \cdot \exp(v_1) \cdot \exp(-v_2) \cdot \exp(-v_1)x_0 = \exp(-v_1) \cdot \{\text{ad}(\exp v_1) \cdot \exp(-v_2)\}x_0 = \exp(-v_1) \cdot \{\exp(-\text{ad}(\exp v_1) \cdot v_2)\}x_0$. Now $\exp(-v_1)$ is an isometry on $G_{\Psi \setminus \Gamma, 0}(x_0)$, and

$$\text{ad}(\exp v_1) \cdot v_2 = \exp(\text{ad } v_1) \cdot v_2 = \sum_{n \geq 0} \frac{1}{n!} \text{ad}(v_1)^n \cdot v_2.$$

Note $\text{ad}(v_1)^n v_2 \in \mathcal{R}_{\Psi \setminus \Gamma, 0}^{\Sigma}$ for n even, $\in \mathcal{Q}_{\Psi \setminus \Gamma}^{\Sigma}$ for n odd; so

$$\text{ad}(\exp v_1) \cdot v_2 = \sum_{n \geq 0} \frac{1}{(2n)!} \text{ad}(v_1)^{2n} \cdot v_2 + \sum_{n \geq 0} \frac{1}{(2n + 1)!} \text{ad}(v_1)^{2n+1} \cdot v_2$$

with the first sum in $\mathcal{R}_{\Psi \setminus \Gamma, 0}^{\Sigma}$ and the second in $\mathcal{Q}_{\Psi \setminus \Gamma}^{\Sigma}$. As $\text{ad}(v_1)$ is symmetric, so $\text{ad}(v_1)^2$ is positive semidefinite, now

$$\|(\text{ad}(\exp v_1) \cdot v_2)_{\mathfrak{M}_{\Psi \setminus \Gamma, 0}}\| \cong \|v_2\|.$$

Thus φ does not decrease lengths of tangent vectors on $v_1 + \mathfrak{R}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}$.

Let $v_2 \in \mathfrak{R}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}$ be fixed, $v_1 \in \mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}$ variable. Then $\varphi(v_1 + v_2) = \exp(-v_2) \exp(-v_1)x_0$ is the composition of -1 on $\mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}$, exponential map in the symmetric space $G_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}(x_0)$ of noncompact type, and left translation by $\exp(-v_2)$. As $G_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}(x_0)$ has curvature ≤ 0 , now φ does not decrease lengths of tangent vectors to $\mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}} + v_2$.

Fix $v_1 \in \mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}$ and let w_1 be variable there; fix $v_2 \in \mathfrak{R}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}$ and let w_2 be variable there. Now

$$\begin{aligned} \varphi(v_1 + \mathfrak{R}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}) &= \{\exp(-v_2 - w_2) \cdot \exp(-v_1) \cdot x_0\}, \\ \varphi(\mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}} + v_2) &= \{\exp(-v_2) \cdot \exp(-v_1 - w_1) \cdot x_0\}. \end{aligned}$$

Those manifolds intersect at $\varphi(v_1 + v_2) = \exp(-v_2) \cdot \exp(-v_1) \cdot x_0$, and we want to show the intersection there to be orthogonal. For that, translate by the isometry $\exp(v_1)$ sending $\exp(-v_2) \exp(-v_1)x_0$ to $\exp(-\text{ad}(\exp v_1) \cdot v_2)x_0 \in \varphi(\mathfrak{R}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}})$, and note first that $\exp(v_1)\varphi(\mathfrak{R}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}})$ consists of all

$$\exp(-\text{ad}(\exp v_1) \cdot v_2 - \text{ad}(\exp v_1)w_2) \cdot x_0$$

and second that $\exp(v_1)\varphi(\mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}})$ consists of all

$$\exp(-\text{ad}(\exp v_1) \cdot v_2) \exp(-w_1) \cdot x_0.$$

Thus the question of orthogonality is reduced to the case $v_1 = 0$, where it follows from first

$$\exp(v_2)\varphi(\mathfrak{R}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}) = \{\exp(-w_2)x_0\} = \varphi(\mathfrak{R}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}})$$

because $\exp(v_2) \exp(-v_2 - w_2) \equiv \exp(-w_2) \pmod{K_{\Psi \setminus \Gamma}^{\mathbb{Z}}}$, and second from

$$\exp(v_2)\varphi(\mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}} + v_2) = \{\exp(-w_1)x_0\} = \varphi(\mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}).$$

Now let ξ be tangent to $\mathfrak{M}_{\Psi \setminus \Gamma, 0}$ at v . Decompose $\xi = \xi_1 + \xi_2$ along $\mathfrak{M}_{\Psi \setminus \Gamma, 0} = \mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}} + \mathfrak{R}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}$. We have proved

$$\|\varphi_*\xi_1\| \cong \|\xi_1\|, \quad \|\varphi_*\xi_2\| \cong \|\xi_2\|, \quad \varphi_*\xi_1 \perp \varphi_*\xi_2.$$

Thus

$$\|\varphi_*\xi\|^2 = \|\varphi_*\xi_1 + \varphi_*\xi_2\|^2 = \|\varphi_*\xi_1\|^2 + \|\varphi_*\xi_2\|^2 \cong \|\xi_1\|^2 + \|\xi_2\|^2 = \|\xi\|^2.$$

That completes the proof of Lemma 11.6. Q.E.D.

We need notation for the factorization of Lemma 11.6. The notation will be, for every $g \in G_{\Psi \setminus \Gamma, 0}$, that

$$(11.7) \quad g = g_K \cdot g_M \cdot g_R \text{ where } g_K \in K_{\Psi \setminus \Gamma}, g_M \in \exp \mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\Sigma}, g_R \in \exp \mathfrak{R}_{\Psi \setminus \Gamma, 0}^{\Sigma}.$$

11.8. THEOREM. *Let Γ and Σ be disjoint subsets of Ψ . Given $k \in K$, define*

$$\beta_k: k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma}) \rightarrow k \cdot K_{\Psi \setminus \Gamma}(x_{\Gamma, \Sigma}) \text{ by } \beta_k(kg_{x_{\Gamma, \Sigma}}) = kg_K(x_{\Gamma, \Sigma}).$$

Now define

$$\beta: G_0(x_{\Gamma, \Sigma}) \rightarrow K(x_{\Gamma, \Sigma}) \text{ by } \beta|_{k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})} = \beta_k.$$

1. $\beta_k: k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma}) \rightarrow k \cdot K_{\Psi \setminus \Gamma}(x_{\Gamma, \Sigma})$ is a well-defined holomorphic fibre bundle with

(1a) structure group: the connected reductive complex Lie group $K_{\Psi \setminus \Gamma}^{\Sigma, \mathbb{C}}$;

(1b) total space: the homomorphic arc component $k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$; of $G_0(x_{\Sigma, \Gamma})$;

(1c) base space: the maximal compact subvariety $k \cdot K_{\Psi \setminus \Gamma}(x_{\Gamma, \Sigma})$ which is a compact totally geodesic hermitian symmetric submanifold of X ;

(1d) fibre over $kk'(x_{\Gamma, \Sigma})$, $k' \in K_{\Psi \setminus \Gamma}$: the totally geodesic hermitian symmetric submanifold

$$\begin{aligned} kk' \cdot G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_{\Gamma, \Sigma}) &= kk' c_{\Gamma \mathbb{C} \Sigma}^2 \cdot G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_0) \\ &= kk' c_{\Gamma \mathbb{C} \Sigma}^2 \{ G_{\Psi \setminus (\Gamma \cup \Sigma), 0}(x_0) \times G_{\Sigma, 0}^{\Sigma}(x_0) \} \end{aligned}$$

of noncompact type and rank $|\Psi \setminus \Gamma|$ in $kk' c_{\Gamma \mathbb{C} \Sigma}^2 \cdot G_0(x_0)$.

2. Let ν be the holomorphic normal bundle to $k \cdot K_{\Psi \setminus \Gamma}(x_{\Gamma, \Sigma})$ in $k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$. Then ν is the homogeneous holomorphic vector bundle over the hermitian symmetric coset space $K_{\Psi \setminus \Gamma} / K_{\Psi \setminus \Gamma}^{\Sigma} \cong k \cdot K_{\Psi \setminus \Gamma}^{\Sigma}(x_{\Gamma, \Sigma})$ defined by the representation of $K_{\Psi \setminus \Gamma}^{\Sigma}$ on the holomorphic tangent space of $G_{\Psi \setminus \Gamma, 0}^{\Sigma} / K_{\Psi \setminus \Gamma}^{\Sigma} \cong G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_{\Gamma, \Sigma})$, and the bundle β_k is holomorphically fibre-equivalent to a relatively compact tubular neighborhood of the zero-section of ν .

3. Let $\det \nu$ denote the determinant bundle of the holomorphic normal bundle ν . If $\dim_{\mathbb{C}} k \cdot K_{\Psi \setminus \Gamma}(x_{\Gamma, \Sigma}) > 0$, then $\det \nu$ is a positive holomorphic line bundle (in the sense of K. Kodaira), and the bundles β_k , ν and $\det \nu$ are all topologically nontrivial. Furthermore, the following conditions are equivalent:

- (3a) β_k is holomorphically trivial;
- (3b) β_k is topologically trivial;
- (3c) β_k has base reduced to a point;
- (3d) $G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$ is hermitian symmetric;

(3e) $\text{ad}(c_z)$ has order 4 on $G_{\Psi \setminus \Gamma}$; and

(3f) If Δ^m are the root systems of the simple ideals of \mathfrak{g} , then for each m , either $\Sigma \cap \Delta^m = \phi$, or $(\Psi \cap \Delta^m) \subset (\Gamma \cup \Sigma)$ with $G_{\Psi \cap \Delta^m, 0}(x_0)$ of tube type.

4. $\beta: G_0(x_{\Gamma, z}) \rightarrow K(x_{\Gamma, z})$ is a well-defined real analytic fibre bundle whose fibres and structure group coincide with those of the restrictions β_k to the holomorphic arc components $k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, z})$ of $G_0(x_{\Gamma, z})$. The bundle β is real analytically equivalent to the homogeneous complex vector bundle over $K(x_{\Gamma, z})$ with fibre equal to the holomorphic tangent space of $G_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}(x_{\Gamma, z})$. The following conditions are equivalent:

(4a) β is real analytically trivial;

(4b) β is topologically trivial;

(4c) for each irreducible factor of X , the corresponding factor of β has base or fibre reduced to a point; and

(4d) if Δ^m are the root systems for the simple ideals of \mathfrak{g} , then for each m , either

(4di) $\Gamma \cap \Delta^m = \phi = \Sigma \cap \Delta^m$, or

(4dii) $\Gamma \cap \Delta^m = \Psi \cap \Delta^m$ and $\Sigma \cap \Delta^m = \phi$, or

(4diii) $\Gamma \cap \Delta^m = \phi$ and $\Sigma \cap \Delta^m = \Psi \cap \Delta^m$ with the bounded domain $G_{\Psi \cap \Delta^m, 0}(x_0)$ of tube type.

PROOF. Fix $k \in K$. Then β_k is well defined by the factorization (11.7) resulting from Lemma 11.6. The total space $k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, z})$ is a holomorphic arc component of $G_0(x_{\Gamma, z})$ by Theorem 10.9, and the base space $kK_{\Psi \setminus \Gamma}(x_{\Gamma, z}) \cong K_{\Psi \setminus \Gamma} / K_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ is a totally geodesic hermitian symmetric submanifold of X by Lemma 11.4. That realizes β_k as a real analytic fibre bundle with structure group $K_{\Psi \setminus \Gamma}^{\mathbb{Z}}$; now we may view $K_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ as the structure group.

Let $k' \in K_{\Psi \setminus \Gamma}$. The β_k -fibre over $kk'(x_{\Gamma, z})$ consists of all $kg(x_{\Gamma, z})$ such that $kk'(x_{\Gamma, z}) = kg_K(x_{\Gamma, z})$. The latter condition is $k'(x_{\Gamma, z}) = g_K(x_{\Gamma, z})$, which is $g_K \in k'K_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ by Lemma 11.4. Thus the fibre consists of all $kk'k_1g_Mg_Rx_{\Gamma, z}$ with $g_M \in \exp \mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}$ and $g_R \in \exp \mathfrak{R}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}$. Lemma 11.5 says $g_Rx_{\Gamma, z} = x_{\Gamma, z}$, and we have $k_1g_Mx_{\Gamma, z} = \text{ad}(k_1)g_Mx_{\Gamma, z}$ which is in $\exp(\mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}})(x_{\Gamma, z})$. Thus the fibre consists of all $kk'g_Mx_{\Gamma, z}$, and now Lemma 11.3 says that it is the totally geodesic hermitian symmetric subspace $kk'G_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}(x_{\Gamma, z}) = kk'c_{\Gamma}c_2^{\mathbb{Z}}G_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}(x_0)$ of noncompact type and rank $|\Psi \setminus \Gamma|$ in $kk'c_{\Gamma}c_2^{\mathbb{Z}}X_0$.

In the notation of [15], it is shown [15, §5.4] that $\mathfrak{g}^{\Psi \setminus \mathbb{Z}} = \{[\mathfrak{O}_1^{\Psi \setminus \mathbb{Z}}, \mathfrak{O}_1^{\Psi \setminus \mathbb{Z}}] + \mathfrak{O}_1^{\Psi \setminus \mathbb{Z}}\} \oplus \mathfrak{L}_2^{\Psi \setminus \mathbb{Z}}$ where the second ideal is in \mathfrak{K} , it is shown [15, Lemma 4.4] that $\mathfrak{O}_1^{\Psi \setminus \mathbb{Z}} = \mathfrak{O}_{\Psi \setminus \mathbb{Z}} + \mathfrak{O}_{\mathbb{Z}, 1}$, and it is shown [15, §6.6] that $[\mathfrak{g}_{\Psi \setminus \mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}, 1}] = 0$. The

- $\mathfrak{G}^{\Psi \setminus \Sigma}$ of [15] is our $\text{ad} \left(\exp \frac{\pi}{4} z \right) \{ \mathfrak{G}_u \cap \mathfrak{G}^{\Sigma} \}$,
- $\mathfrak{P}_1^{\Psi \setminus \Sigma}$ of [15] is our $\text{ad} \left(\exp \frac{\pi}{4} z \right) \{ \mathfrak{M}_u \cap \mathfrak{G}^{\Sigma} \}$,
- $\mathfrak{P}_{\Psi \setminus \Sigma}$ of [15] is our $\text{ad} \left(\exp \frac{\pi}{4} z \right) \{ \mathfrak{M}_u \cap \mathfrak{G}_{\Psi \setminus \Sigma} \}$,
- $\mathfrak{P}_{\Sigma, 1}$ of [15] is our $\text{ad} \left(\exp \frac{\pi}{4} z \right) \{ \mathfrak{M}_u \cap \mathfrak{G}^{\Sigma} \}$,
- $\mathfrak{G}_{\Psi \setminus \Sigma}$ of [15] is our $\text{ad} \left(\exp \frac{\pi}{4} z \right) \{ \mathfrak{G}_u \cap \mathfrak{G}_{\Psi \setminus \Sigma} \}$,
- $\mathfrak{G}_{\Sigma, 1}$ of [15] is defined as $[\mathfrak{P}_{\Sigma, 1}, \mathfrak{P}_{\Sigma, 1}] + \mathfrak{P}_{\Sigma, 1}$.

Using that dictionary we conclude that

$$\mathfrak{G}_{\Psi}^{\Sigma} = \mathfrak{G}_{\Psi \setminus \Sigma} \oplus \mathfrak{G}_{\Sigma}^{\Sigma} \oplus \mathfrak{L}_{\Sigma}, \quad \mathfrak{L}_{\Sigma} \subset \mathcal{K}^C.$$

Taking intersection with $\mathfrak{G}_{\Psi \setminus \Gamma}$, now

$$\mathfrak{G}_{\Psi \setminus \Gamma}^{\Sigma} = \mathfrak{G}_{\Psi \setminus (\Gamma \cup \Sigma)} \oplus \mathfrak{G}_{\Sigma}^{\Sigma} \oplus \mathfrak{L}_{\Gamma, \Sigma}, \quad \mathfrak{L}_{\Gamma, \Sigma} \subset \mathcal{K}^C.$$

All three ideals are τ -stable. Thus

$$\mathfrak{G}_{\Psi \setminus \Gamma, 0}^{\Sigma} = \mathfrak{G}_{\Psi \setminus (\Gamma \cup \Sigma), 0} \oplus \mathfrak{G}_{\Sigma, 0}^{\Sigma} \oplus \mathfrak{L}_{\Gamma, \Sigma, 0}, \quad \mathfrak{L}_{\Gamma, \Sigma, 0} \subset \mathcal{K}.$$

It follows that

$$G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_0) = G_{\Psi \setminus (\Gamma \cup \Sigma), 0}(x_0) \times G_{\Sigma, 0}^{\Sigma}(x_0).$$

We have proved every part of (1) except for the fact that the bundle β_k is holomorphic. However we have seen that its total space, base and fibres are complex submanifolds of X , and that its structure group is a complex Lie group acting holomorphically on the typical fibre, so we need only demonstrate the existence of holomorphic transition functions. Thus, to complete the proof of (1) it suffices to complete the proof of (2).

$G_{\Psi \setminus \Gamma, 0}(x_0)$ has holomorphic tangent space $\mathfrak{M}_{\Psi \setminus \Gamma} = \mathfrak{M}^- \cap \mathfrak{M}_{\Psi \setminus \Gamma, 0}^C$ at x_0 , and $\mathfrak{M}_{\Psi \setminus \Gamma}^{\Sigma} = \mathfrak{M}^- \cap \mathfrak{M}_{\Psi \setminus \Gamma, 0}^{\Sigma C}$ is the subspace that is the holomorphic tangent space to $G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_0)$. Define $\mathfrak{R}_{\Psi \setminus \Gamma}^{\Sigma} = \mathfrak{M}^- \cap \mathfrak{R}_{\Psi \setminus \Gamma, 0}^{\Sigma C}$, so $\mathfrak{M}_{\Psi \setminus \Gamma} = \mathfrak{M}_{\Psi \setminus \Gamma}^{\Sigma} + \mathfrak{R}_{\Psi \setminus \Gamma}^{\Sigma}$. Recalling that $\text{ad}(c_{\Sigma}^2)$ interchanges $\mathfrak{R}_{\Psi \setminus \Gamma, 0}^{\Sigma C}$ with $\mathfrak{Q}_{\Psi \setminus \Gamma}^{\Sigma C}$ and noting that $\text{ad}(c_{\Sigma}^4) \mathfrak{R}_{\Psi \setminus \Gamma}^{\Sigma} = \mathfrak{R}_{\Psi \setminus \Gamma}^{\Sigma}$, we define $\mathfrak{Q}_{\Psi \setminus \Gamma}^{\Sigma} = \text{ad}(c_{\Sigma}^2) \mathfrak{R}_{\Psi \setminus \Gamma}^{\Sigma}$; then $\text{ad}(c_{\Sigma}^2) \mathfrak{M}_{\Psi \setminus \Gamma} = \text{ad}(c_{\Sigma}^2) \mathfrak{M}_{\Psi \setminus \Gamma}^{\Sigma} + \mathfrak{Q}_{\Psi \setminus \Gamma}^{\Sigma}$. Now Lemmas 11.3, 11.4 and 11.5 say that, at $x_{\Gamma, \Sigma}$,

$G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \mathbb{Z}})$ has holomorphic tangent space $\text{ad}(c_{\mathbb{Z}}^2) \mathfrak{M}_{\Psi \setminus \Gamma}^{\mathbb{Z}-} + \mathbb{Q}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}$,
 $G_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}(x_{\Gamma, \mathbb{Z}})$ has holomorphic tangent space $\text{ad}(c_{\mathbb{Z}}^2) \mathfrak{M}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}$,
 $K_{\Psi \setminus \Gamma}(x_{\Gamma, \mathbb{Z}})$ has holomorphic tangent space $\mathbb{Q}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}$.

Now the holomorphic normal space of $K_{\Psi \setminus \Gamma}(x_{\Gamma, \mathbb{Z}})$ in $G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \mathbb{Z}})$ at $x_{\Gamma, \mathbb{Z}}$ is $\text{ad}(c_{\mathbb{Z}}^2) \mathfrak{M}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}$, and $K_{\Psi \setminus \Gamma}^{\mathbb{Z}C}$ preserves it and acts there by its representation on the holomorphic tangent space to $G_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}(x_{\Gamma, \mathbb{Z}})$. That identifies the holomorphic normal bundle ν of $k \cdot K_{\Psi \setminus \Gamma}(x_{\Gamma, \mathbb{Z}})$ in $k \cdot G_{\Psi \setminus \Gamma}(x_{\Gamma, \mathbb{Z}})$ with the homogeneous vector bundle over $K_{\Psi \setminus \Gamma} / K_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ with fibre $\text{ad}(c_{\mathbb{Z}}^2) \mathfrak{M}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}$ that has $K_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ -module structure from ad_g . The Harish-Chandra realization of $G_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}(x_{\Gamma, \mathbb{Z}})$ as a bounded domain in $\text{ad}(c_{\mathbb{Z}}^2) \mathfrak{M}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}$ identifies β_k with a relatively compact open sub-bundle of ν . Now (2) is proved, and that completes the proof of (1).

Let κ be the canonical (holomorphic) line bundle over $k \cdot K_{\Psi \setminus \Gamma}(x_{\Gamma, \mathbb{Z}})$. It is the holomorphic line bundle whose sections are the $(d, 0)$ -forms, $d = \dim_{\mathbb{C}} k \cdot K_{\Psi \setminus \Gamma}(x_{\Gamma, \mathbb{Z}})$. Thus its dual κ^* has fibre $\Lambda^d \mathbb{Q}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}$ where of course $d = \dim_{\mathbb{C}} \mathbb{Q}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}$. Let δ denote the representation of $K_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ on the fibre of κ , so its dual δ^* is the representation on $\Lambda^d \mathbb{Q}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}$. Define $z_{\Psi \setminus \Gamma} = (i/2) \sum_{\Psi \setminus \Gamma} h_{\Psi} \in \mathfrak{K}_{\Psi \setminus \Gamma}^{\mathbb{Z}}$. Then [15, Lemma 6.2] $\text{ad}(z_{\Psi \setminus \Gamma})$ acts on $\mathfrak{R}_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ as scalar multiplication by $i/2$. As $\text{ad}(c_{\mathbb{Z}}^2)$ preserves $\mathfrak{K}_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ and interchanges $\mathfrak{R}_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ with $\mathbb{Q}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}$, now $\text{ad}(c_{\mathbb{Z}}^2)(2z_{\Psi \setminus \Gamma})$ is an element of $\mathfrak{K}_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ that acts on $\mathbb{Q}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}$ under ad_g as scalar multiplication by i . Thus

$$z_{\Psi \setminus \Gamma}^{\mathbb{Z}} = \text{ad}(c_{\mathbb{Z}}^2)(2z_{\Psi \setminus \Gamma}) \text{ is central in } \mathfrak{K}_{\Psi \setminus \Gamma}^{\mathbb{Z}}$$

and

$$\text{ad}(z_{\Psi \setminus \Gamma}^{\mathbb{Z}}) \Big|_{\mathbb{Q}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}}^{\mathbb{Z}C} \text{ is the almost complex structure of } k \cdot K_{\Psi \setminus \Gamma}(x_{\Gamma, \mathbb{Z}}).$$

Let $\mathfrak{E} \subset \mathfrak{K}_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ denote the Lie algebra of the kernel of the action of $K_{\Psi \setminus \Gamma}$ on $K_{\Psi \setminus \Gamma}(x_{\Gamma, \mathbb{Z}}) \cong K_{\Psi \setminus \Gamma} / K_{\Psi \setminus \Gamma}^{\mathbb{Z}}$. Now δ^* can be described by: $\delta^*(z_{\Psi \setminus \Gamma}) = \text{id}$, $\delta^*(\mathfrak{E}) = 0$, and δ^* annihilates the orthocomplement of the image of $z_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ in $\mathfrak{K}_{\Psi \setminus \Gamma}^{\mathbb{Z}} / \mathfrak{E}$. Thus we describe κ as the homogeneous line bundle on $K_{\Psi \setminus \Gamma} / K_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ induced by the character δ on $K_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ such that $\delta(z_{\Psi \setminus \Gamma}^{\mathbb{Z}}) = -\text{id}$, $\delta(\mathfrak{E}) = 0$, and δ annihilates the orthocomplement of the image of $z_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ in $\mathfrak{K}_{\Psi \setminus \Gamma}^{\mathbb{Z}} / \mathfrak{E}$.

Let ν over $k \cdot K_{\Psi \setminus \Gamma}(x_{\Gamma, \mathbb{Z}})$ denote the holomorphic normal bundle in $k \cdot G_{\Psi \setminus \Gamma}(x_{\Gamma, \mathbb{Z}})$. We will describe the determinant bundle $\det \nu$ over $k \cdot K_{\Psi \setminus \Gamma}(x_{\Gamma, \mathbb{Z}})$ so as to be able to compare it with the canonical bundle κ . $\det \nu$ has fibre $\Lambda^a \text{ad}(c_{\mathbb{Z}}^2) \mathfrak{M}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}$ where $a = \dim_{\mathbb{C}} \mathfrak{M}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}$. Let α denote the representation of $K_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ on that fibre. We know [15, Lemma 4.4] that $\text{ad}(z_{\Psi \setminus \Gamma})$ is scalar multiplication by i on $\mathfrak{M}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}$; thus $\text{ad}(z_{\Psi \setminus \Gamma}^{\mathbb{Z}})$ is scalar multiplication by $2i$ on $\text{ad}(c_{\mathbb{Z}}^2) \mathfrak{M}_{\Psi \setminus \Gamma}^{\mathbb{Z}-}$; we conclude that $\alpha(z_{\Psi \setminus \Gamma}^{\mathbb{Z}})$

$= 2ia$. Let $\mathcal{E}' \subset \mathcal{K}_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ denote the Lie algebra of the kernel of the action of $G_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}$ on $G_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}(x_{\Gamma, \mathbb{Z}}) \cong G_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}/K_{\Psi \setminus \Gamma}^{\mathbb{Z}}$. Then \mathcal{E}' is orthogonal to the subalgebra $\mathfrak{G}_0[\Psi \setminus \Gamma]$ of $\mathfrak{G}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}$ that contains $z_{\Psi \setminus \Gamma}^{\mathbb{Z}}$. Thus we can specify: $\alpha(z_{\Psi \setminus \Gamma}^{\mathbb{Z}}) = 2ia$ and α annihilates the orthocomplement of $z_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ in $\mathcal{K}_{\Psi \setminus \Gamma}^{\mathbb{Z}}$. Comparing representations we now conclude that $(\det \nu)^d = \kappa^{-2a}$ as holomorphic line bundles.

Recall $d = \dim_{\mathbb{C}} k \cdot K_{\Psi \setminus \Gamma}(x_{\Gamma, \mathbb{Z}})$ and $a = \dim_{\mathbb{C}} G_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}(x_{\Gamma, \mathbb{Z}})$. If $a = 0$, then the noncompact type symmetric space $G_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}(x_{\Gamma, \mathbb{Z}})$ is reduced to a point, so its rank $|\Psi \setminus \Gamma| = 0$; then $\Gamma = \Psi$ so $G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \mathbb{Z}}) = x_{\Gamma, \mathbb{Z}} = K_{\Psi \setminus \Gamma}(x_{\Gamma, \mathbb{Z}})$, whence $d = 0$. In other words, if $d > 0$, then $a > 0$.

Suppose $d > 0$. Then the canonical bundle κ of $K_{\Psi \setminus \Gamma}/K_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ is negative in the sense of K. Kodaira; as $a > 0$ it follows that κ^{-2a} is positive; thus $\det \nu$ is a positive line bundle. Let $[c(\det \nu)] \in H^{1,1}(K_{\Psi \setminus \Gamma}/K_{\Psi \setminus \Gamma}^{\mathbb{Z}}; \mathbf{Z})$ denote the Chern class of $\det \nu$. Now $[c(\det \nu)]$ is represented by a $K_{\Psi \setminus \Gamma}$ -invariant positive definite (1, 1)-form $c(\det \nu)$. As it is positive definite, $c(\det \nu) \neq 0$; as it is $K_{\Psi \setminus \Gamma}$ -invariant and $K_{\Psi \setminus \Gamma}/K_{\Psi \setminus \Gamma}^{\mathbb{Z}}$ is symmetric, now $[c(\det \nu)] \neq 0$; thus $\det \nu$ is not topologically trivial. It follows that ν is not topologically trivial. As the bundles ν and β_k are topologically equivalent, now β_k is not topologically trivial.

If $d = 0$, then $K_{\Psi \setminus \Gamma} = K_{\Psi \setminus \Gamma}^{\mathbb{Z}}$, so $\mathcal{Q}_{\Psi \setminus \Gamma}^{\mathbb{Z}} = 0$; then $\mathcal{R}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}} = \text{ad}(c_{\mathbb{Z}}) \cdot (i\mathcal{Q}_{\Psi \setminus \Gamma}^{\mathbb{Z}}) = 0$, so $\mathfrak{G}_{\Psi \setminus \Gamma, 0} = \mathfrak{G}_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}$. That says that $\text{ad}(c_{\mathbb{Z}})$ has order 4 on $\mathfrak{G}_{\Psi \setminus \Gamma}$. Let $\mathfrak{g} = \sum \mathfrak{g}^m$ decomposition into simple ideals, $\Delta = \cup \Delta^m$ corresponding decomposition of the root system of \mathfrak{g} . Then [15, Corollary 4.10] the $\mathfrak{g}_{\Psi \setminus \Gamma}^m = \mathfrak{g}_{(\Psi \setminus \Gamma) \cap \Delta^m}$ are the simple ideals of $\mathfrak{G}_{\Psi \setminus \Gamma}$ (possibly with some of them equal to zero); further [15, Corollary 4.11] if $G_{(\Psi \setminus \Gamma) \cap \Delta^m, 0}(x_0)$ is of tube type and positive dimension, then $G_{\Psi \cap \Delta^m, 0}(x_0)$ is of tube type; finally [15, Theorem 5.7] $\text{ad}(c_{\mathbb{Z}})$ has order 4 on $\mathfrak{G}_{\Psi \setminus \Gamma}$ if and only if, for each simple ideal $\mathfrak{g}_{(\Psi \setminus \Gamma) \cap \Delta^m}$ of $\mathfrak{G}_{\Psi \setminus \Gamma}$, either

- (i) $\Sigma \cap (\Psi \setminus \Gamma) \cap \Delta^m$ is empty or
- (ii) $G_{(\Psi \setminus \Gamma) \cap \Delta^m, 0}(x_0)$ is of tube type and $(\Psi \setminus \Gamma) \cap \Delta^m \subset \Sigma$.

We conclude, using disjointness of Γ and Σ , that $\text{ad}(c_{\mathbb{Z}})$ has order 4 on $\mathfrak{G}_{\Psi \setminus \Gamma}$ if and only if, for each m , either

- (i) $\Sigma \cap \Delta^m = \phi$ or
- (ii) $G_{\Psi \cap \Delta^m, 0}(x_0)$ is of tube type and $\Psi \cap \Delta^m \subset (\Gamma \cup \Sigma)$.

That completes the proof of (3).

We will prove $\beta: G_0(x_{\Gamma, \mathbb{Z}}) \rightarrow K(x_{\Gamma, \mathbb{Z}})$ well-defined. In other words, given $k, k' \in K$ with $k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \mathbb{Z}}) = k' \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \mathbb{Z}})$, we will prove $\beta_k = \beta_{k'}$. Now suppose

$$x \in k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \mathbb{Z}}) = k' \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \mathbb{Z}}); \quad k, k' \in K.$$

Express

$$x = kg(x_{\Gamma, \mathbb{Z}}) = k'g'(x_{\Gamma, \mathbb{Z}}); \quad g, g' \in G_{\Psi \setminus \Gamma, 0}.$$

$k^{-1}k' \in K$ preserves the holomorphic arc component $G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$ of $G_0(x_{\Gamma, \Sigma})$; that holomorphic arc component being a symmetric space of noncompact type, now $k^{-1}k' = k_1 k_2$ where $k_1 \in K_{\Psi \setminus \Gamma}$ and k_2 acts trivially on $G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$. Thus we may ignore k_2 and assume

$$k' = k k_1 \quad \text{with } k_1 \in K_{\Psi \setminus \Gamma}.$$

As in (11.7) we decompose

$$g = g_K g_M g_R \quad \text{and} \quad g' = g'_K g'_M g'_R.$$

Using $g_R x_{\Gamma, \Sigma} = x_{\Gamma, \Sigma} = g'_R x_{\Gamma, \Sigma}$, now we have

$$x = k g_K g_M x_{\Gamma, \Sigma} = k k_1 g'_K g'_M x_{\Gamma, \Sigma}.$$

Thus $g_K g_M h = k_1 g'_K g'_M$ where $h \in G_{\Psi \setminus \Gamma, 0}$ fixes $x_{\Gamma, \Sigma}$. Lemma 11.5 says $h = h_K h_R$, $h_K \in K_{\Psi \setminus \Gamma}^{\Sigma}$ and $h_R \in \exp \mathfrak{R}_{\Psi \setminus \Gamma, 0}^{\Sigma}$. Now

$$(k_1 g'_K) g'_M = g_K g_M h_K h_R = (g_K h_K) (h_K^{-1} g'_M h_K) h_R.$$

Uniqueness of (11.7) thus says

$$k_1 g'_K = g_K h_K, \quad g'_M = h_K^{-1} g'_M h_K, \quad 1 = h_R.$$

Thus

$$\beta_k(x) = k g_K(x_{\Gamma, \Sigma}) = k g_K h_K(x_{\Gamma, \Sigma}) = k k_1 g'_K(x_{\Gamma, \Sigma}) = k' g'_K(x_{\Gamma, \Sigma}) = \beta_{k'}(x),$$

and β is proved to be well defined.

Now $\beta: G_0(x_{\Gamma, \Sigma}) \rightarrow K(x_{\Gamma, \Sigma})$ is a well-defined real analytic map. If $x \in G_0(x_{\Gamma, \Sigma})$, then x and $\beta(x)$ are in the same holomorphic arc component of $G_0(x_{\Gamma, \Sigma})$. It follows that β is a real analytic bundle whose fibres coincide with those of the restrictions β_k . The structure group of β is the isotropy subgroup of K at $x_{\Gamma, \Sigma}$. That isotropy group preserves the holomorphic arc component $G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$, so its every element has factorization $k = k_1 k_2$ with $k_1 \in G_{\Psi \setminus \Gamma, 0} \cap K = K_{\Psi \setminus \Gamma}$ and k_2 acting trivially on the fibre $G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$ of β . Note $k_1 \in K_{\Psi \setminus \Gamma}^{\Sigma}$ because $k_1(x_{\Gamma, \Sigma}) = x_{\Gamma, \Sigma}$. Thus we can reduce the structure group of β to the structure group $K_{\Psi \setminus \Gamma}^{\Sigma}$ of its restrictions β_k .

Consider the special case $\beta: G_0(x_{\Gamma, \phi}) \rightarrow K(x_{\Gamma, \phi})$ where G_0 is simple and $\phi \neq \Gamma \neq \Psi$. The fibre of β over $k(x_{\Gamma, \phi})$ is the holomorphic arc component $k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \phi})$ of $G_0(x_{\Gamma, \phi})$. The space $K(x_{\phi, \Gamma})$, maximal compact subvariety of the open orbit $G_0(x_{\phi, \Gamma})$, has been seen to be a totally geodesic hermitian symmetric subspace of compact type in X . We will construct a positive holomorphic line bundle λ over $K(x_{\phi, \Gamma})$, and a fibration $f: K(x_{\Gamma, \phi}) \rightarrow K(x_{\phi, \Gamma})$, such that

$f^*\lambda$ is the determinant bundle of $\beta: G_0(x_{\Gamma,\phi}) \rightarrow K(x_{\Gamma,\phi})$.

For that construction we use the notation

L^Γ : isotropy subgroup of K at $x_{\Gamma,\phi}$;

K^Γ : isotropy subgroup of K at $x_{\phi,\Gamma}$.

Lemma 11.4 says that $K^\Gamma = K_{\Psi}^\Gamma$, centralizer of c_Γ^4 in K ; the decomposition $\mathfrak{G}_{\Psi,\phi}^\Gamma = \mathfrak{L}_{\Gamma,\phi,0} \oplus \mathfrak{G}_{\Psi \setminus \Gamma,0} + \mathfrak{G}_{\Gamma,0}^\Gamma$ informs us first that $L^\Gamma = L_{\Gamma,\phi,0} \cdot K_{\Psi \setminus \Gamma}$ local direct product where the first factor acts trivially on $G_{\Psi \setminus \Gamma,0}(x_{\Gamma,\phi})$, and second that $K^\Gamma = L^\Gamma \cdot K_\Gamma^\Gamma$ local direct product. Now we define

$$f: K(x_{\Gamma,\phi}) = K/L^\Gamma \rightarrow K/K^\Gamma = K(x_{\phi,\Gamma}) \quad \text{by } f(kL^\Gamma) = kK^\Gamma.$$

View f as identification under the right action of K_Γ^Γ on K/L^Γ . That action commutes with the action of L^Γ on the fibres of β . Thus, in the category of topological vector bundles,

$$\beta = f^*\gamma \quad \text{for some vector bundle } \gamma \text{ over } K(x_{\phi,\Gamma}).$$

As K^Γ acts on the fibres $k \cdot G_{\Psi \setminus \Gamma,0}(x_{\Gamma,\phi})$ of γ via the action of $K_{\Psi \setminus \Gamma}^C$ on $G_{\Psi \setminus \Gamma,0}(x_{\Gamma,\phi})$, which is holomorphic,

γ over $K(x_{\phi,\Gamma})$ is the K -homogeneous holomorphic

vector bundle with fibre $\mathfrak{N}_{\Psi \setminus \Gamma}^-$ over $x_{\phi,\Gamma}$.

Now we compare $\det \gamma$ with the canonical bundle κ over $K(x_{\phi,\Gamma})$. $\text{ad}(c_\Gamma^2)z$ acts on the fibre $\Lambda^a \mathfrak{N}_{\Psi \setminus \Gamma}^-$, $a = \dim_C \mathfrak{N}_{\Psi \setminus \Gamma}^-$, of $\det \gamma$, as multiplication by ia . It acts on the fibre $(\Lambda^d \mathfrak{Q}_{\Psi}^\Gamma)^* = (\Lambda^d \text{ad}(c_\Gamma^2) \mathfrak{R}_{\Psi}^\Gamma)^*$, $d = \dim_C \mathfrak{Q}_{\Psi}^\Gamma = \dim_C K(x_{\phi,\Gamma})$, of κ , as multiplication by $-id$. Thus as before,

$$(\det \gamma)^d = \kappa^{-a}.$$

Our hypothesis that G_0 be simple and $\phi \neq \Gamma \neq \Psi$ ensures $d \neq 0 \neq a$. As κ is negative, now $(\det \gamma)^d$ is positive, so $\det \gamma$ is positive. Define $\lambda = \det \gamma$, and we have

$$f^*\lambda = f^* \det \gamma = \det f^*\gamma = \det(\beta: G_{\Psi \setminus \Gamma,0}(x_{\Gamma,\phi}) \rightarrow K(x_{\Gamma,\phi})).$$

That completes the construction.

Suppose G_0 simple. We will prove that $\beta: G_0(x_{\Gamma,\Sigma}) \rightarrow K(x_{\Gamma,\Sigma})$ is topologically trivial if and only if

(i) $\Gamma = \phi = \Sigma$, so β is fibration of $G_0(x_0) = X_0$ over the single point x_0 , or

(ii) $\Gamma = \Psi$ and $\Sigma = \phi$, so $\beta: G_0(x_{\Psi,\phi}) \rightarrow K(x_{\Psi,\phi})$ is the fibration of the Bergman-Šilov boundary of X_0 in X over itself with fibre reduced to a point; or

(iii) $\Gamma = \phi$, $\Sigma = \Psi$, and $X_0 = G_0(x_0)$ is of tube type, so β is fibration of $G_0(x_{\phi, \Psi}) = c_{\Psi}^2 X_0$ over a single point $x_{\phi, \Psi} = c_{\Psi}^2 x_0$.

First note that β is topologically trivial in the three cases described just above. Conversely, now assume β topologically trivial. Then the restrictions β_k to the holomorphic arc components $k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$ of $G_0(x_{\Gamma, \Sigma})$ are topologically trivial, so either $\Sigma = \phi$, or $\Psi = \Gamma \cup \Sigma$ with X_0 of tube type. First suppose $\Sigma = \phi$, so $\beta: G_0(x_{\Gamma, \phi}) \rightarrow K(x_{\Gamma, \phi})$, and recall the map $f: K(x_{\Gamma, \phi}) \rightarrow K(x_{\phi, \Gamma})$ of the paragraph before last. If $\phi \neq \Gamma \neq \Psi$, we saw that the determinant bundle of β , viewing β topologically as a complex vector bundle, is of the form $f^* \lambda$ where λ over $K(x_{\phi, \Gamma})$ is a positive holomorphic line bundle. As its Chern class is nonzero, λ is not topologically trivial, so $f^* \lambda = \det \beta$ is not topologically trivial, contradicting topological triviality of β . Thus either $\Gamma = \phi$ and we are in case (i) above, or $\Gamma = \Psi$ and we are in case (ii) above. Now suppose $\Sigma \neq \phi$, so $\Psi = \Gamma \cup \Sigma$ with X_0 of tube type. As X_0 is of tube type, i.e. as $c_{\Psi}^4 = 1$, we have K and G_0 stable under $\text{ad}(c_{\Psi}^2)$. Now, using the consequence $c_{\Psi} = c_{\Gamma} c_{\Sigma}$ of $\Psi = \Gamma \cup \Sigma$ and $\Gamma \cap \Sigma = \phi$, we have

$$c_{\Psi}^2: G_0(x_{\Gamma, \Sigma}) \rightarrow G_0(c_{\Psi}^2 x_{\Gamma, \Sigma}) = G_0(c_{\Gamma}^3 c_{\Sigma}^4 x_0) = G_0(x_{\Gamma, \phi}),$$

and that the map has the properties

$$c_{\Psi}^2 K(x_{\Gamma, \Sigma}) = K(x_{\Gamma, \phi}) \quad \text{and} \quad c_{\Psi}^2 k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma}) = k' \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \phi}).$$

Thus c_{Ψ}^2 defines a fibre bundle isomorphism between the fibrations β given by

$$G_0(x_{\Gamma, \Sigma}) \rightarrow K(x_{\Gamma, \Sigma}) \quad \text{and} \quad G_0(x_{\Gamma, \phi}) \rightarrow K(x_{\Gamma, \phi}).$$

The first being topologically trivial by hypothesis, the second must be trivial, and we have just seen that to imply either $\Gamma = \phi$ or $\Gamma = \Psi$. But $\Psi = \Gamma \cup \Sigma$, $\Gamma \cap \Sigma = \phi$ and $\Sigma \neq \phi$; thus $\Gamma = \phi$, $\Sigma = \Psi$ and we are in case (iii) above.

Let \mathfrak{g}^m denote the simple ideals of \mathfrak{g} , Δ^m the root system of \mathfrak{g}^m , $\mathfrak{g}_0^m = \mathfrak{g}_0 \cap \mathfrak{g}^m$, G^m and G_0^m the analytic subgroups for \mathfrak{g}^m and \mathfrak{g}_0^m , and $X^m = G^m(x_0)$ and $X_0^m = G_0^m(x_0)$. Then $G_0(x_{\Gamma, \Sigma})$ is the product of the $G_0^m(x_{\Gamma \cap \Delta^m, \Sigma \cap \Delta^m})$, and the base and fibres of $\beta: G_0(x_{\Gamma, \Sigma}) \rightarrow K(x_{\Gamma, \Sigma})$ decompose similarly. Thus β is topologically trivial if and only if each of the "factors"

$$G_0^m(x_{\Gamma \cap \Delta^m, \Sigma \cap \Delta^m}) \rightarrow (K \cap G_0^m)(x_{\Gamma \cap \Delta^m, \Sigma \cap \Delta^m})$$

is topologically trivial. According to the paragraph before the last one, β is topologically trivial if and only if, for each m ,

- (i) $\Psi \cap \Delta^m = \phi = \Sigma \cap \Delta^m$, or

- (ii) $\Gamma \cap \Delta^m = \Psi \cap \Delta^m$ and $\Sigma \cap \Delta^m = \phi$, or
- (iii) $\Gamma \cap \Delta^m = \phi$ and $\Sigma \cap \Delta^m = \Psi \cap \Delta^m$ with X_0^m of tube type.

That completes the proof of (4), completing the proof of Theorem 11.8. Q.E.D.

11.9. COROLLARY. *Let $\Gamma \subset \Psi$. Then the fibration $\beta: G_0(x_{\Gamma, \phi}) \rightarrow K(x_{\Gamma, \phi})$ of Theorem 11.8 is given by $\beta(k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \phi})) = k(x_{\Gamma, \phi})$ for $k \in K$, and it is real analytically equivalent to a complex homogeneous vector bundle μ over $K(x_{\Gamma, \phi})$. There is a fibration $f: K(x_{\Gamma, \phi}) \rightarrow K(x_{\phi, \Gamma})$ defined by $f(kx_{\Gamma, \phi}) = kx_{\phi, \Gamma}$, whose fibres are compact totally geodesic submanifolds $k \cdot K_{\Psi}^{\Gamma}(x_{\Gamma, \phi})$ of X , and whose base is the totally geodesic hermitian symmetric subspace $K(x_{\phi, \Gamma})$ of X ; $\mu = f^* \gamma$ for a certain holomorphic homogeneous vector bundle γ over $K(x_{\phi, \Gamma})$. If $\phi \neq \Gamma \cap \Delta^m \neq \Psi \cap \Delta^m$ where Δ^m is the root system of some simple ideal of \mathfrak{g} , then $\dim_{\mathbb{C}} K(x_{\phi, \Gamma}) > 0$ and $\det \gamma$ is a positive homogeneous holomorphic line bundle over $K(x_{\phi, \Gamma})$; in particular, in that case $\det \gamma, \gamma, \det \mu, \mu$ and β all are topologically nontrivial.*

11.10. COROLLARY. *Let $\Gamma \subset \Psi$ and $\Sigma = \Psi \setminus \Gamma$ with X_0 of tube type. Then the fibration $\beta: G_0(x_{\Gamma, \Sigma}) \rightarrow K(x_{\Gamma, \Sigma})$ of Theorem 11.8 is given by mapping $c_{\Psi}^2 k G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma}) = \text{ad}(c_{\Psi}^2) k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$ to $c_{\Psi}^2 kx_{\Gamma, \Sigma} = \text{ad}(c_{\Psi}^2) k \cdot x_{\Gamma, \Sigma}$, and it is real analytically equivalent to a complex homogeneous vector bundle μ' over $K(x_{\Gamma, \Sigma})$. There is a fibration $f': K(x_{\Gamma, \Sigma}) \rightarrow K(x_{\phi, \Sigma})$ defined by $f'(kx_{\Gamma, \Sigma}) = kx_{\phi, \Sigma}$, whose fibres are compact totally geodesic submanifolds $k \cdot K_{\Psi}^{\Gamma}(x_{\Gamma, \Sigma}) = k \cdot K_{\Psi}^{\Sigma}(x_{\Gamma, \Sigma})$ of X , and whose base is the totally geodesic hermitian symmetric subspace $K(x_{\phi, \Sigma})$ of X ; $\mu' = f'^* \gamma'$ where γ' is a certain holomorphic homogeneous vector bundle over $K(x_{\phi, \Sigma})$. If $\phi \neq \Sigma \cap \Delta^m \neq \Psi \cap \Delta^m$ where Δ^m is the root system of some simple ideal of \mathfrak{g} , then $\dim_{\mathbb{C}} K(x_{\phi, \Sigma}) > 0$ and $\det \gamma'$ is a positive homogeneous holomorphic line bundle over $K(x_{\phi, \Sigma})$; in particular, in that case $\det \gamma', \gamma', \det \mu', \mu'$ and β are all topologically nontrivial.*

Corollaries 10.9 and 10.10 are related by: c_{Ψ}^2 is a bundle isomorphism of $G_0(x_{\Gamma, \Sigma}) \rightarrow K(x_{\Gamma, \Sigma})$ onto $G_0(x_{\Gamma, \phi}) \rightarrow K(x_{\Gamma, \phi})$ and maps $K(x_{\phi, \Sigma})$ onto $K(x_{\phi, \Gamma})$, inducing the fibrations f' from f, μ' from μ and γ' from γ .

PROOF OF COROLLARIES. Corollary 11.9 is contained in the proof of part 4 of Theorem 11.8 for the case where \mathfrak{g} is simple, and the general case follows. The second paragraph of the statement of Corollary 11.10 is also contained in that proof, and the first paragraph then follows from Corollary 11.9. Q.E.D.

The special case of an open orbit is worth separate examination.

11.11. COROLLARY. Every open G_0 -orbit on X has a holomorphic fibration

$$\beta: G_0(x_{\phi, \Sigma}) \rightarrow K(x_{\phi, \Sigma}) \text{ by } \beta(gx_{\phi, \Sigma}) = g_K x_{\phi, \Sigma}$$

where

$$g = g_K g_M g_R, \quad g_K \in K, \quad g_M \in \exp \mathfrak{M}_{\Psi, 0}^{\Sigma}, \quad g_R \in \exp \mathfrak{R}_{\Psi, 0}^{\Sigma}.$$

1. The bundle β has structure group K_{Ψ}^{Σ} , has total space $G_0(x_{\phi, \Sigma})$, has base space $K(x_{\phi, \Sigma})$ which is a compact totally geodesic hermitian symmetric submanifold of X , and has fibre over $k(x_{\phi, \Sigma})$ which is the totally geodesic hermitian symmetric submanifold

$$k \cdot G_{\Psi, 0}^{\Sigma}(x_{\phi, \Sigma}) = kc_{\Sigma}^2 G_{\Psi, 0}^{\Sigma}(x_0) = kc_{\Sigma}^2 \{G_{\Psi \cap \Sigma, 0}(x_0) \times G_{\Sigma, 0}^{\Sigma}(x_0)\}$$

of noncompact type and rank $|\Psi|$ in $kc_{\Sigma}^2 G_0(x_0)$.

2. The holomorphic normal ν bundle of $K(x_{\phi, \Sigma})$ in $G_0(x_{\phi, \Sigma})$ is the homogeneous vector bundle over $K(x_{\phi, \Sigma}) \cong K/K_{\Psi}^{\Sigma}$ associated to the representation of K_{Ψ}^{Σ} on the holomorphic tangent space of $G_{\Psi, 0}^{\Sigma}/K_{\Psi}^{\Sigma}$. The bundle β is holomorphically equivalent to a relatively compact open subbundle of that holomorphic normal bundle ν . If $\dim_{\mathbb{C}} K(x_{\phi, \Sigma}) > 0$, i.e. if the root system Δ^m of some simple ideal of \mathfrak{g} satisfies $\Sigma \cap \Delta^m \neq \emptyset$, and satisfies $\Sigma \cap \Delta^m \neq \Psi \cap \Delta^m$ in case $G_{\Psi \cap \Delta^m, 0}(x_0)$ is of tube type, then $\det \nu$ is a positive homogeneous holomorphic line bundle over $K(x_{\phi, \Sigma})$, and the bundles $\det \nu$, ν , and β all are topologically nontrivial.

PROOF. This is the case $\Gamma = \phi$ of Theorem 11.8. Q.E.D.

The case of the "opposite" open orbit is of further interest because it gives new characterizations of tube domains.

11.12. COROLLARY. The "opposite" open orbit $G_0(x_{\phi, \Psi}) \subset X$ has a holomorphic fibration

$$\beta: G_0(x_{\phi, \Psi}) \rightarrow K(x_{\phi, \Psi}) \text{ by } \beta(gx_{\phi, \Psi}) = g_K x_{\phi, \Psi}$$

where

$$g = g_K g_M g_R, \quad g_K \in K, \quad g_M \in \exp \mathfrak{M}_{\Psi, 0}^{\Psi}, \quad g_R \in \exp \mathfrak{R}_{\Psi, 0}^{\Psi}.$$

1. The bundle β has structure group K_{Ψ}^{Ψ} , total space $G_0(x_{\phi, \Psi})$, base space $K(x_{\phi, \Psi})$ which is a maximal compact subvariety of $G_0(x_{\phi, \Psi})$ and is a compact totally geodesic hermitian symmetric submanifold of X , and has fibre over $k(x_{\phi, \Psi})$ which is the "tube part" (c.f. [2])

$$k \cdot G_{\Psi, 0}^{\Psi}(x_{\phi, \Psi}) = kc_{\Psi}^2 \cdot G_{\Psi, 0}^{\Psi}(x_0)$$

of $kc_{\Psi}^2 \cdot G_0(x_0)$.

2. The bundle β is a relatively compact open subbundle of the holomorphic normal bundle ν of $K(x_{\phi, \Psi})$ in $G_0(x_{\phi, \Psi})$. ν is the homogeneous holomorphic vector bundle over $K(x_{\phi, \Psi}) = K/K_{\Psi}^{\mathbb{Z}}$ associated to the representation of $K_{\Psi}^{\mathbb{Z}}$ on the holomorphic tangent space of the tube part $G_{\Psi, 0}^{\mathbb{Z}}(x_0)$ of $G_0(x_0)$.

3. The following conditions are equivalent:

- (3a) β is holomorphically trivial.
- (3b) ν is holomorphically trivial.
- (3c) $\det \nu$ is holomorphically trivial.
- (3d) β is topologically trivial.
- (3e) ν is topologically trivial.
- (3f) $\det \nu$ is topologically trivial.
- (3g) $K(x_{\phi, \Psi})$ is the single point $x_{\phi, \Psi}$.
- (3h) c_{Ψ} has order 4 on X .

(3i) The bounded symmetric domain $X_0 = G_0(x_0)$ is of tube type, i.e. it has a holomorphic realization as a tube domain over a homogeneous self dual cone.

(3j) The Bergman-Silov boundary $G_0(x_{\Psi, \phi})$ of X_0 in X is a totally geodesic submanifold of X .

(3k) $\dim_{\mathbb{R}} G_0(x_{\Psi, \phi}) = \dim_{\mathbb{C}} X$.

(3l) Viewing G_0 as the identity component of the group $G_{\mathbb{R}}$ of real points in the linear algebraic group G , we have $P_{x_{\Psi, \phi}}$ defined/ \mathbb{R} , i.e. X is defined/ \mathbb{R} .

PROOF. Statements 1 and 2, and equivalence of (3a) through (3h), are the case $\Sigma = \Psi$ of Corollary 11.11. Equivalence of (3h), (3i), (3j) and (3k) is contained in [2, Theorems 4.9 and 6.8], and Theorem 3.6 says that (3k) is equivalent to (3l). Q.E.D.

We recapitulate the decomposition of an arbitrary G_0 -orbit $G_0(x_{\Gamma, \Sigma}) \subset X$. Here Γ and Σ are disjoint subsets of Ψ . First we have the fibration of Theorem 11.8,

$$\beta: G_0(x_{\Gamma, \Sigma}) \rightarrow K(x_{\Gamma, \Sigma}),$$

that maps an arbitrary holomorphic arc component $k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$ holomorphically onto its maximal compact subvariety $k \cdot K_{\Psi \setminus \Gamma}(x_{\Gamma, \Sigma})$; the fibre over $kk'(x_{\Gamma, \Sigma})$, $k \in K$ and $k' \in K_{\Psi \setminus \Gamma}$, is

$$kk' \cdot G_{\Psi \setminus \Gamma, 0}^{\mathbb{Z}}(x_{\Gamma, \Sigma}) = kk' c_{\Gamma} c_{\Sigma}^{\mathbb{Z}} \{ G_{\Psi \setminus (\Gamma \cup \Sigma), 0}(x_0) \times G_{\Sigma, 0}^{\mathbb{Z}}(x_0) \}.$$

We then follow β by $\alpha \cdot \beta^{-1}$ where

$$\alpha: G_0(x_{\Gamma, \Sigma}) \rightarrow G_0/N_{\Gamma, \Sigma, 0}$$

is the fibration of the orbit over the space of its holomorphic arc components. This amounts to collapsing each of the maximal compact subvarieties $k \cdot K_{\Psi \setminus \Gamma}(x_{\Gamma, \Sigma}) \subset k \cdot G_{\Psi \setminus \Gamma, 0}(x_{\Gamma, \Sigma})$ to a point. That collapsing is, of course, already done when, whenever Δ^m is the root system of a simple ideal of \mathfrak{g} , either $\Sigma \cap \Delta^m = \phi$, or $(\Gamma \cup \Sigma) \cap \Delta^m = \Psi \cap \Delta^m$ with $G_{\Psi \cap \Delta^m, 0}(x_0)$ of tube type. Now we come to the final stage of the decomposition, which is realization of the β -fibre

$$G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_{\Gamma, \Sigma}) = G_{\Psi \setminus (\Gamma \cup \Sigma), 0}(x_{\Gamma, \Sigma}) \times G_{\Sigma, 0}^{\Sigma}(x_{\Gamma, \Sigma})$$

as a Siegel domain of type III over any of its boundary components. That Siegel domain realization is the content of Wolf-Korányi [15, §7]; the appendix to this §11 transcribes it into our notation, and explicitly works out the holomorphic fibration over the boundary component (Proposition 11 A.3) that is implicit in the Siegel domain formulation, for the hermitian symmetric space $X_0 = G_0(x_0)$. We now recast the results of that appendix to this §11 to cover our case of the hermitian symmetric space

$$\begin{aligned} G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_{\Gamma, \Sigma}) &= G_{\Psi \setminus (\Gamma \cup \Sigma), 0}(x_{\Gamma, \Sigma}) \times G_{\Sigma, 0}^{\Sigma}(x_{\Gamma, \Sigma}) \\ &= c_{\Gamma} c_{\Sigma}^2 \{ G_{\Psi \setminus (\Gamma \cup \Sigma), 0}(x_0) \times G_{\Sigma, 0}^{\Sigma}(x_0) \} = c_{\Gamma} c_{\Sigma}^2 G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_0). \end{aligned}$$

$\Psi \setminus \Gamma$ is the maximal set of strongly orthogonal noncompact positive roots of $\mathfrak{g}_{\Psi \setminus \Gamma, 0}^{\Sigma}$. Let E be any subset. Then we have disjoint unions $\Psi \setminus (\Gamma \cup \Sigma) = A \cup A'$, $\Sigma = B \cup B'$, $E = A \cup B$, $E' = A' \cup B'$.

Now observe

$$\begin{aligned} \mathfrak{N}_{\Psi \setminus \Gamma}^{\Sigma^-} \cap \mathfrak{N}_E^{B^-} &= \mathfrak{N}_A^{A^-} + \mathfrak{N}_B^{B^-}, \\ \mathfrak{N}_{\Psi \setminus \Gamma}^{E^-} \cap \mathfrak{R}_{\Psi}^{E^-} &= \mathfrak{R}_{\Psi \setminus (\Gamma \cup \Sigma)}^{A^-} + (\mathfrak{R}_{\Sigma}^{B^-} \cap \mathfrak{N}_{\Sigma}^{\Sigma^-}), \text{ and} \\ \mathfrak{N}_{\Psi \setminus \Gamma}^{\Sigma^-} \cap \mathfrak{N}_{\Psi \setminus E}^{-} &= \mathfrak{N}_A^{-} + \mathfrak{N}_{B'}^{B'^-}. \end{aligned}$$

Thus we define

$$(11.13a) \quad V'_1 = \mathfrak{N}_A^{A^-}, \quad V'_2 = \mathfrak{R}_{\Psi \setminus (\Gamma \cup \Sigma)}^{A^-} \text{ and } V'_3 = \mathfrak{N}_A^{-};$$

$$(11.13b) \quad V''_1 = \mathfrak{N}_B^{B^-}, \quad V''_2 = (\mathfrak{R}_{\Sigma}^{B^-} \cap \mathfrak{N}_{\Sigma}^{\Sigma^-}) \text{ and } V''_3 = \mathfrak{N}_B^{B'^-}; \text{ and}$$

$$(11.13c) \quad V_1 = V'_1 \oplus V''_1, \quad V_2 = V'_2 \oplus V''_2 \text{ and } V_3 = V'_3 \oplus V''_3.$$

Now every $v \in \mathfrak{N}_{\Psi \setminus \Gamma}^{\Sigma^-}$ has unique decomposition

$$(11.13d) \quad v = v'_1 + v''_1 + v'_2 + v''_2 + v'_3 + v''_3, \quad v'_i \in V'_i \text{ and } v''_j \in V''_j$$

that we use without further explanation.

If $P \subset Q \subset \Psi$ we have the bounded domain $D_Q^P = \xi^{-1}G_{Q,0}^P(x_0)$ in \mathfrak{M}_Q^P . Now the bounded domains in V_3', V_3'' and V_3 that serve our Siegel domains, are

$$(11.14a) \quad D_{A'} = \xi^{-1}G_{A',0}(x_0) \subset V_3', \quad D_{B'}^{B'} = \xi^{-1}G_{B',0}^{B'}(x_0) \subset V_3'', \quad \text{and}$$

$$(11.14b) \quad D_{A'} \times D_{B'}^{B'} = D_{E'}^{B'} = \xi^{-1}G_{E',0}^{B'}(x_0) \subset V_3.$$

Note that they are the respective intersections of V_3', V_3'' and V_3 with the bounded domain $D_{\Psi \setminus E,0}(x_0) \subset \mathfrak{M}_{\Psi \setminus E}$.

The real forms of V_1', V_1'' and V_1 are

$$(11.15a) \quad U_1' = \text{ad}(c_A)\{\mathfrak{G}_0 \cap \text{ad}(c_A^{-1})V_1'\},$$

$$U_1'' = \text{ad}(c_B)\{\mathfrak{G}_0 \cap \text{ad}(c_B^{-1})V_1''\} \quad \text{and}$$

$$(11.15b) \quad U_1' \oplus U_1'' = U_1 = \text{ad}(c_E)\{\mathfrak{G}_0 \cap \text{ad}(c_E^{-1})V_1\}.$$

They contain the self dual cones

$$(11.16a) \quad \mathfrak{C}_A = \text{ad}(K_A^{A*}) \cdot \xi^{-1}(x_{A,\phi}) \subset U_1' \quad \text{where} \quad \xi^{-1}(x_{A,\phi}) = \sum_{A \in \mathfrak{e}-\mathfrak{p}},$$

$$(11.16b) \quad \mathfrak{C}_B = \text{ad}(K_B^{B*}) \cdot \xi^{-1}(x_{B,\phi}) \subset U_1'' \quad \text{where} \quad \xi^{-1}(x_{B,\phi}) = \sum_{B \in \mathfrak{e}-\mathfrak{p}},$$

$$(11.16c) \quad \mathfrak{C}_A \times \mathfrak{C}_B = \mathfrak{C}_E = \text{ad}(K_E^{E*}) \cdot \xi^{-1}(x_{E,\phi}) \subset U_1 \quad \text{where}$$

$$\xi^{-1}(x_{E,\phi}) = \sum_{E \in \mathfrak{e}-\mathfrak{p}}.$$

If $w \in D_{E'}^{B'}$ we have the conjugate-linear map $\mu(w): V_2 \rightarrow V_2$ of operator norm < 1 , given by

$$(11.17a) \quad \mu(w)v = \text{ad}(w) \cdot \text{ad}(c_E^2) \cdot \tau_u(v) = \text{ad}(w') \cdot \text{ad}(c_A^2) \cdot \tau_u(v') \\ + \text{ad}(w'') \cdot \text{ad}(c_B^2) \cdot \tau_u(v'').$$

It specifies the "bilinear plus hermitian" map $F_w: V_2 \times V_2 \rightarrow V_1$ by

$$(11.17b) \quad F_w(u, v) = - (i/2) [u, \text{ad}(c_E^2) \cdot \tau_u \cdot (I - \mu(w))^{-1} v].$$

The splitting in (11.17a) says that $\mu(w): V_2 \rightarrow V_2$ is the direct sum of operators

$$\mu(w'): V_2' \rightarrow V_2' \quad \text{and} \quad \mu(w''): V_2'' \rightarrow V_2''.$$

Thus $(I - \mu(w))^{-1}: V_2 \rightarrow V_2$ splits as the direct sum of

$$(I - \mu(w'))^{-1}: V_2' \rightarrow V_2' \quad \text{and} \quad (I - \mu(w''))^{-1}: V_2'' \rightarrow V_2''.$$

It follows that $F_w: V_2 \times V_2 \rightarrow V_1$ is the direct sum of maps

$$F_{w'}: V_2' \times V_2' \rightarrow V_1' \quad \text{and} \quad F_{w''}: V_2'' \times V_2'' \rightarrow V_1''$$

in the sense that, for $w \in D_{E'}^{B'}$ and $u, v \in V_2$,

$$(11.17c) \quad F_w(u, v) = F_{w'}(u', v') + F_{w''}(u'', v'').$$

Carrying (11A.1) back to our situation, now

$$(11.18a) \quad \mathfrak{S}_{\Psi \setminus (\Gamma \cup \Sigma), A} = \{v' \in \mathfrak{M}_{\Psi \setminus (\Gamma \cup \Sigma)}^- : v'_3 \in D_{A'} \text{ and} \\ \text{Im } v'_1 - F_{v'_3}(v'_2, v'_2) \in \mathfrak{C}_{A'}\},$$

$$(11.18b) \quad \mathfrak{S}_{\Sigma, B}^{\Sigma} = \{v'' \in \mathfrak{M}_{\Sigma}^{\Sigma-} : v''_3 \in D_{B'} \text{ and} \\ \text{Im } v''_1 = F_{v''_3}(v''_2, v''_2) \in \mathfrak{C}_{B'}\},$$

$$(11.18c) \quad \mathfrak{S}_{\Psi \setminus \Gamma, E}^{\Sigma} = \{v \in \mathfrak{M}_{\Psi \setminus \Gamma}^{\Sigma-} : v_3 \in D_{E'} \text{ and} \\ \text{Im } v_1 - F_{v_3}(v_2, v_2) \in \mathfrak{C}_{E'}\}$$

are Siegel domains of type III that satisfy

$$(11.18d) \quad \mathfrak{S}_{\Psi \setminus \Gamma, E}^{\Sigma} = \mathfrak{S}_{\Psi \setminus (\Gamma \cup \Sigma), A} \times \mathfrak{S}_{\Sigma, B}^B \subset \mathfrak{M}_{\Psi \setminus (\Gamma \cup \Sigma)}^- \oplus \mathfrak{M}_{\Sigma}^{\Sigma-} = \mathfrak{M}_{\Psi \setminus \Gamma}^{\Sigma-}.$$

Furthermore, (11A.2) in our context says that

$$(11.19a) \quad c_A G_{\Psi \setminus (\Gamma \cup \Sigma), 0}(x_0) \subset \xi(\mathfrak{M}_{\Psi \setminus (\Gamma \cup \Sigma)}^-) \text{ and} \\ \xi^{-1} : c_A G_{\Psi \setminus (\Gamma \cup \Sigma), 0}(x_0) \cong \mathfrak{S}_{\Psi \setminus (\Gamma \cup \Sigma), A},$$

$$(11.19b) \quad c_B G_{\Sigma, 0}^{\Sigma}(x_0) \subset \xi(\mathfrak{M}_{\Sigma}^{\Sigma-}) \text{ and } \xi^{-1} : c_B G_{\Sigma, 0}^{\Sigma}(x_0) \cong \mathfrak{S}_{\Sigma, B}^{\Sigma},$$

$$(11.19c) \quad c_E G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_0) \subset \xi(\mathfrak{M}_{\Psi \setminus \Gamma}^{\Sigma-}) \text{ and } \xi^{-1} : c_E G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_0) \cong \mathfrak{S}_{\Psi \setminus \Gamma, E}^{\Sigma},$$

such that the analytic equivalence of (11.19c) is the direct product of those of (11.19a) and (11.19b). Finally, now, the coarse structure portion of Proposition 11A.3 becomes:

11.20. PROPOSITION. *Let Γ and Σ be disjoint subsets of Ψ , let $E \subset \Psi \setminus \Gamma$, and define disjoint union decompositions by*

$$E = A \cup B, \quad E' = A' \cup B', \quad \Psi \setminus (\Gamma \cup \Sigma) = A \cup A' \text{ and } \Sigma = B \cup B'.$$

Retain the notation (11.13) through (11.18). Then we have holomorphic fibre bundles

$$\pi' : \mathfrak{S}_{\Psi \setminus (\Gamma \cup \Sigma), A} \rightarrow D_{A'}, \quad \pi'' : \mathfrak{S}_{\Sigma, B}^{\Sigma} \rightarrow D_{B'}^{B'} \text{ and } \pi : \mathfrak{S}_{\Psi \setminus \Gamma, E}^{\Sigma} \rightarrow D_{E'}^{B'}$$

given by

$$\pi'(v') = v'_3, \quad \pi''(v'') = v''_3 \text{ and } \pi(v) = v_3 = v'_3 + v''_3.$$

1. *There are connected Lie groups $H_{\pi'}$, $H_{\pi''}$ and H_{π} that act transitively on the respective total spaces of π' , π'' and π by holomorphic bundle*

automorphisms. They have semidirect product structure

$$H_{\pi'} = G_{A',0} \cdot Z_{\pi'}, \quad H_{\pi''} = G_{B',0} \cdot Z_{\pi''} \quad \text{and} \quad H_{\pi} = G_{E',0} \cdot Z_{\pi}$$

where the first factor is equivariant for its natural transitive action on the base, and where the normal subgroups composing the second factor are the respective identity components of

$$G_{\Psi \setminus (\Gamma \cup \Sigma)} \cap \text{ad}(c_A)Z_{A,\phi,0}, \quad G_{\Sigma}^2 \cap \text{ad}(c_B)Z_{B,\phi,0} \quad \text{and} \quad G_{\Psi \setminus \Gamma}^2 \cap \text{ad}(c_E)Z_{E,\phi,0}.$$

The latter are trivial on the base and transitive on every fibre. Finally, H_{π} acts on $\mathfrak{S}_{\Psi \setminus \Gamma, E}^2 = \mathfrak{S}_{\Psi \setminus (\Gamma \cup \Sigma), A} \times \mathfrak{S}_{\Sigma, B}^2$ as $H_{\pi'} \times H_{\pi''}$.

2. The fibres of π' , π'' and π are holomorphically equivalent to hermitian symmetric spaces of noncompact type with respective ranks $|A|$, $|B|$ and $|E| = |A| + |B|$ and with respective tube parts $G_{A,0}^A(x_0)$, $G_{B,0}^B(x_0)$ and $G_{E,0}^E(x_0) = G_{A,0}^A(x_0) \times G_{B,0}^B(x_0)$. The realizations of the fibres over 0 as Siegel domains of type II are

$$\begin{aligned} \pi'^{-1}(0) &= \{v'_1 + v'_2 \in \mathfrak{M}_A^{A-} + \mathfrak{R}_{\Psi \setminus (\Gamma \cup \Sigma)}^{A-} : \text{Im } v'_1 - F_0(v'_2, v'_2) \in \mathfrak{C}_A\}, \\ \pi''^{-1}(0) &= v''_1 + v''_2 \in \mathfrak{M}_B^{B-} + (\mathfrak{R}_{\Sigma}^{B-} \cap \mathfrak{M}_{\Sigma}^{2-}) : \\ &\quad \text{Im } v''_1 - F_0(v''_2, v''_2) \in \mathfrak{C}_B\}, \\ \pi^{-1}(0) &= \pi'^{-1}(0) \times \pi''^{-1}(0) = \{v \in V_1 \oplus V_2 : \text{Im } v_1 - F_0(v_2, v_2) \in \mathfrak{C}_E\}. \end{aligned}$$

3. The bundles π' , π'' and π are real-analytically trivial. Let $\{\Delta'_k\}$ be the root systems of the simple ideals of $\mathfrak{S}_{\Psi \setminus (\Gamma \cup \Sigma)}$, $\{\Delta''_i\}$ the root systems of the simple ideals of \mathfrak{S}_{Σ}^2 . Then

(3a) π' is holomorphically trivial if and only if A is a union of sets of the form $[\Psi \setminus (\Gamma \cup \Sigma)] \cap \Delta'_k$,

(3b) π'' is holomorphically trivial if and only if B is a union of sets of the form $\Sigma \cap \Delta''_i$, and

(3c) π is holomorphically trivial if and only if both π' and π'' are holomorphically trivial.

We come to the final step of the decomposition of $G_0(x_{\Gamma, \Sigma})$. Recall $\beta: G_0(x_{\Gamma, \Sigma}) \rightarrow K(x_{\Gamma, \Sigma})$ with $\beta^{-1}k(x_{\Gamma, \Sigma}) = k \cdot G_{\Psi \setminus \Gamma, 0}^2(x_{\Gamma, \Sigma})$. Proposition 11.20 fibres

$$(11.21a) \quad \beta^{-1}(x_{\Gamma, \Sigma}) = G_{\Psi \setminus \Gamma, 0}^2(x_{\Gamma, \Sigma}) \rightarrow G_{E', 0}^{B'}(x_{\Gamma, \Sigma})$$

by

$$(11.21b) \quad c_{\Gamma} c_{\Sigma} c_B^{-1} \xi(v) \rightarrow c_{\Gamma} c_{\Sigma} \xi(\pi v) \quad \text{for every } v \in \mathfrak{S}_{\Psi \setminus \Gamma, E}^2.$$

As seen from the case where the unit disc is fibered over one of its (one point) boundary components, $G_{E',0}^{B'}$ ($x_{\Gamma,\Sigma}$) is generally not $K_{\Psi\setminus\Gamma}^{\Sigma}$ -stable, so we cannot just extend (11.21) to $G_0(x_{\Gamma,\Sigma})$ by K -equivariance. That problem is avoided by use of a measurable, a.e. real analytic, "section" of the principal $K_{\Psi\setminus\Gamma}^{\Sigma}$ -bundle $K \rightarrow K/K_{\Psi\setminus\Gamma}^{\Sigma}$.

Let $\mathcal{U}_{\Psi\setminus\Gamma}^{\Sigma}$ denote the orthocomplement of $\mathfrak{K}_{\Psi\setminus\Gamma}^{\Sigma}$ in \mathfrak{K} ; define $V_{\Psi\setminus\Gamma}^{\Sigma} = \exp(\mathcal{U}_{\Psi\setminus\Gamma}^{\Sigma}) \subset K$. Then every $k \in K$ has expression $k = vk'$, $v \in V_{\Psi\setminus\Gamma}^{\Sigma}$ and $k' \in K_{\Psi\setminus\Gamma}^{\Sigma}$, but that factorization need not be unique. To approach uniqueness, let $K/K_{\Psi\setminus\Gamma}^{\Sigma}$ carry a riemannian metric induced by a bi-invariant metric on K , so the exponential map

$$\exp_{\theta}: \mathcal{U}_{\Psi\setminus\Gamma}^{\Sigma} \rightarrow K/K_{\Psi\setminus\Gamma}^{\Sigma} \quad \text{by } \exp_{\theta}(v) = \exp(v) \cdot \theta$$

where θ is the base point. Now let $\mathfrak{W}_{\Psi\setminus\Gamma}^{\Sigma}$ consist of all $v \in \mathcal{U}_{\Psi\setminus\Gamma}^{\Sigma}$ such that $\exp_{\theta}(tv)$, $0 \leq t \leq 1$, is the unique minimizing geodesic from θ to $\exp_{\theta}(v)$. Define $W_{\Psi\setminus\Gamma}^{\Sigma} = \exp(\mathfrak{W}_{\Psi\setminus\Gamma}^{\Sigma}) \subset V_{\Psi\setminus\Gamma}^{\Sigma}$. Then

(11.22a) $W_{\Psi\setminus\Gamma}^{\Sigma}(\theta)$ is a dense open subset of $K/K_{\Psi\setminus\Gamma}^{\Sigma}$, and

(11.22b) $\exp_{\theta}: \mathfrak{W}_{\Psi\setminus\Gamma}^{\Sigma} \rightarrow W_{\Psi\setminus\Gamma}^{\Sigma}(\theta)$ is a diffeomorphism.

That gives us a "section" $s: K/K_{\Psi\setminus\Gamma}^{\Sigma} \rightarrow K$ by

(11.22c) $s(\exp_{\theta}(w)) = \exp(w) \in W_{\Psi\setminus\Gamma}^{\Sigma}$ for $w \in \mathfrak{W}_{\Psi\setminus\Gamma}^{\Sigma}$,

(11.22d) $s(k \cdot \theta) \in kK_{\Psi\setminus\Gamma}^{\Sigma}$ arbitrary for $k \cdot \theta \notin W_{\Psi\setminus\Gamma}^{\Sigma}(\theta)$.

Although s may be discontinuous, it is real analytic on the dense open subset $W_{\Psi\setminus\Gamma}^{\Sigma}(\theta) \subset K/K_{\Psi\setminus\Gamma}^{\Sigma}$ whose complement has measure zero.

11.23. THEOREM. *Let Γ and Σ be disjoint subsets of Ψ , let $E \subset \Psi \setminus \Gamma$, and consider disjoint unions*

$$E = A \cup B, \quad E' = A' \cup B', \quad \Psi \setminus (\Gamma \cup \Sigma) = A \cup A', \quad \Sigma = B \cup B'.$$

If $k \in K$, define

$$\gamma_k^{A,B}: k \cdot G_{\Psi\setminus\Gamma,0}^{\Sigma}(x_{\Gamma,\Sigma}) \rightarrow k \cdot G_{E',0}^{B'}(x_{\Gamma,\Sigma})$$

by

$$\gamma_k^{A,B}(k c_{\Gamma} c_{\Sigma}^{-1} \xi v) = k c_{\Gamma} c_{\Sigma}^2 \xi(\tau v) \quad \text{for } v \in \mathfrak{G}_{\Psi\setminus\Gamma,E}^{\Sigma}.$$

Retain the notation (11.22) and define

$$\gamma^{A,B}: G_0(x_{\Gamma,\Sigma}) \rightarrow s(K/K_{\Psi\setminus\Gamma}^{\Sigma}) \cdot G_{E',0}^{B'}(x_{\Gamma,\Sigma})$$

by

$$\gamma^{A,B} \Big|_{kG_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_{\Gamma, \Sigma})} = \gamma_{s(k \cdot \Theta)}^{A,B}.$$

1. Given $k \in K$, $\gamma_k^{A,B}$ is a holomorphic fibre bundle that is equivalent to the bundle $\pi: \mathfrak{S}_{\Psi \setminus \Gamma, E}^{\Sigma} \rightarrow D_{E'}^{B'}$ of Proposition 11.20 under the maps:

$$\begin{array}{ccc} k \cdot G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_{\Gamma, \Sigma}) & \xrightarrow{\xi^{-1} c_E c_{\Sigma}^{-2} c_{\Gamma}^{-1} k^{-1}} & \mathfrak{S}_{\Psi \setminus \Gamma, E}^{\Sigma} \\ \downarrow \gamma_k^{A,B} & & \downarrow \pi \\ k \cdot G_{E', 0}^{B'}(x_{\Gamma, \Sigma}) & \xrightarrow{\xi^{-1} c_{\Sigma}^{-2} c_{\Gamma}^{-1} k^{-1}} & D_{E'}^{B'} \end{array}$$

2. $\gamma^{A,B}$ is a well-defined, but not necessarily continuous, "fibre bundle" whose fibres coincide with those of its restrictions $\gamma_{s(k \cdot \Theta)}^{A,B}$, and which is real analytic over $W_{\Psi \setminus \Gamma}^{\Sigma} \cdot G_{E', 0}^{B'}(x_{\Gamma, \Sigma})$.

3. The fibration $\beta: G_0(x_{\Gamma, \Sigma}) \rightarrow K(x_{\Gamma, \Sigma})$ of Theorem 11.8 comes from $\gamma^{A,B}$ by

$$\beta = \{ \beta \cdot (\gamma^{A,B})^{-1} \} \cdot \gamma^{A,B} \quad \text{where} \quad \{ \beta \cdot (\gamma^{A,B})^{-1} \} \cdot k G_{E'}^{B'}(x_{\Gamma, \Sigma}) = k(x_{\Gamma, \Sigma}).$$

PROOF. Statement (1) is immediate from Proposition 11.20. $\gamma_k^{A,B}$ sends the β -fibre $k \cdot G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_{\Gamma, \Sigma}) = s(k \cdot \Theta) \cdot G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_{\Gamma, \Sigma})$ over $k(x_{\Gamma, \Sigma}) = s(k \cdot \Theta)(x_{\Gamma, \Sigma})$ into its hermitian symmetric subspace $k \cdot G_{E', 0}^{B'}(x_{\Gamma, \Sigma})$. As β is well defined, (3) follows.

We check that $\gamma^{A,B}$ is well defined. Let $x \in G_0(x_{\Gamma, \Sigma})$. Suppose $\beta(x) = k(x_{\Gamma, \Sigma})$, $k \in K$. Then $x \in \beta^{-1}k(x) = s(k \cdot \Theta) \cdot G_{\Psi \setminus \Gamma, 0}^{\Sigma}(x_{\Gamma, \Sigma})$, so $x = s(k \cdot \Theta)g(x_{\Gamma, \Sigma})$ for some $g \in G_{\Psi \setminus \Gamma, 0}^{\Sigma}$. Now, by definition, $\gamma^{A,B}(x) = \gamma_{s(k \cdot \Theta)}^{A,B}(x) = s(k \cdot \Theta)c_{\Gamma}c_{\Sigma}\xi(\pi v)$ where $v \in \mathfrak{S}_{\Psi \setminus \Gamma, E}^{\Sigma}$ such that $g(x_{\Gamma, \Sigma}) = c_{\Gamma}c_{\Sigma}^2c_E^{-1}\xi(v)$. That procedure is unique, first because k is specified modulo $K_{\Psi \setminus \Gamma}$ because β is well defined, and then because k is specified modulo $K_{\Psi \setminus \Gamma}^{\Sigma}$ by Lemma 11.4.

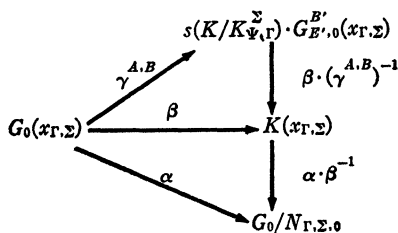
$(\gamma^{A,B})^{-1}(s(k \cdot \Theta)(x_{\Gamma, \Sigma})) = (\gamma_{s(k \cdot \Theta)}^{A,B})^{-1}(s(k \cdot \Theta)(x_{\Gamma, \Sigma}))$ by construction and well definition of $\beta \cdot (\gamma^{A,B})^{-1}$.

Over $W_{\Psi \setminus \Gamma}^{\Sigma} \cdot G_{E', 0}^{B'}(x_{\Gamma, \Sigma})$, $\gamma^{A,B}$ is given by

$$\gamma^{A,B}(kc_{\Gamma}c_{\Sigma}^2c_E^{-1}\xi(v)) = kc_{\Gamma}c_{\Sigma}^2\xi(\pi v), \quad k \in W_{\Psi \setminus \Gamma}^{\Sigma} \quad \text{and} \quad v \in \mathfrak{S}_{\Psi \setminus \Gamma, E}^{\Sigma}.$$

That is real analytic. Q.E.D.

We now have the triple fibration



of an arbitrary orbit $G_0(x_{\Gamma, \Sigma}) \subset X$ as described at the beginning of this §11.

APPENDIX TO §11—SIEGEL DOMAINS. We first recall the concepts of cone and Siegel domain.

Let U be a real vector space with a positive definite inner product $\langle \cdot, \cdot \rangle$. Then a *self dual cone* in U means a subset $\mathfrak{C} \subset U$ such that

$$\mathfrak{C} = \{u \in U: \text{if } 0 \neq v \in \mathfrak{C} \text{ then } \langle u, v \rangle > 0\}.$$

It is automatic that a self dual cone is a cone (nonempty, and $u \in \mathfrak{C}$ and $r > 0$ imply $ru \in \mathfrak{C}$), is open, is convex, and does not contain a straight line. "Domain of positivity" is another term for self dual cone.

Let $\mathfrak{C} \subset U$ be a convex open cone that does not contain a straight line. Define $V = U + iU$ complexification; if $v \in V$ then $v = (\text{Re } v) + i(\text{Im } v)$ with $\text{Re } v$ and $\text{Im } v$ in U . The **tube domain** over \mathfrak{C} is defined to be the set $\{v \in V: \text{Im } v \in \mathfrak{C}\}$. The Poincaré and Siegel upper half planes are examples. By **Siegel domain of type I** we mean a tube domain over a convex open cone that does not contain a straight line. A hermitian symmetric space is of **tube type** if it is holomorphically diffeomorphic to a Siegel domain of type I; in that case the cone is self dual. A standard classification result says that a hermitian symmetric space is of tube type if and only if it is a product, each of whose factors is one of

- $SU(m, m)/S(U(m) \times U(m)), \quad m \geq 1; \text{ or}$
- $SO^*(4m)/U(2m), \quad m \geq 2; \text{ or}$
- $Sp(m; \mathbb{R})/U(m), \quad m \geq 1; \text{ or}$
- $SO(2, m)/SO(2) \times SO(m), \quad m \geq 1; \text{ or}$
- $E_{7, \text{index}=25}/E_6 \cdot SO(2).$

Let V_1 and V_2 be complex vector spaces, U_1 a real form of V_1 , $\mathfrak{C} \subset U_1$ a cone as above, and $F: V_2 \times V_2 \rightarrow V_1$ an hermitian map that is positive definite in the sense

$$0 \neq v_2 \in V_2 \quad \text{implies } 0 \neq F(v_2, v_2) \in \bar{\mathbb{C}}.$$

That data defines the domain

$$\{(v_1, v_2) \in V_1 \oplus V_2: \text{Im } v_1 - F(v_2, v_2) \in \bar{\mathbb{C}}\}.$$

Those domains are the **Siegel domains of type II**. Siegel domains of type I are the special case $V_2=0$. Every hermitian symmetric space of noncompact type has a standard realization (Korányi-Wolf [2]) as a Siegel domain of type II, and that Siegel domain is of type I if and only if the hermitian symmetric space is of tube type; the cone is self dual.

Let V_1, V_2 and V_3 be complex vector spaces, U_1 a real form of V_1 , $\mathbb{C} \subset U_1$ a cone as above, and $D \subset V_3$ a bounded domain. Suppose further that, for every $w \in D$, we have

$$F_w^h: V_2 \times V_2 \rightarrow V_1 \text{ hermitian and}$$

$$F_w^b: V_2 \times V_2 \rightarrow V_1 \text{ bilinear}$$

depending smoothly on $w \in D$, such that

$$F_0^b = 0 \quad \text{and} \quad F_0^h \text{ is positive definite relative to } \mathbb{C}.$$

Then we define

$$F_w: V_2 \times V_2 \rightarrow V_1 \quad \text{by } F_w = F_w^h + F_w^b.$$

Assume that for every $w \in D$, F_w is nondegenerate in the sense that each of $F_w(v, V_2) = 0$ and $F_w(V_2, v) = 0$ implies $v = 0$. Then the domain

$$\{(v_1, v_2, v_3) \in V_1 \oplus V_2 \oplus V_3: v_3 \in D \text{ and } \text{Im } v_1 - \text{Re } F_{v_3}(v_2, v_2) \in \bar{\mathbb{C}}\}$$

is called a **Siegel domain of type III** with **base** D . Siegel domains of type II are the special case $V_3=0$, i.e. the case where the base is reduced to a point.

In [15, §7] Wolf and Korányi describe $X_0 = G_0(x_0)$ as a Siegel domain of type III in \mathfrak{M}^- whose base is the bounded domain in $\mathfrak{M}_{\Psi \setminus E}^-$ equivalent to an arbitrary boundary component $G_{\Psi \setminus E, 0}(x_E, \phi)$ of X_0 . That description involves the holomorphic injection

$$\xi: \mathfrak{M}^- \rightarrow X \quad \text{by } \xi(v) = \exp(v) \cdot x_0$$

of \mathfrak{M}^- as a dense open subset of X , and is given in our notation as follows.

First note $\mathfrak{M}^- = \mathfrak{M}_{\Psi}^{E-} + \mathfrak{R}_{\Psi}^{E-} = \mathfrak{M}_{\Psi \setminus E}^- + \mathfrak{M}_E^{E-} + \mathfrak{R}_{\Psi}^{E-}$ direct. Now if $v \in \mathfrak{M}^-$, we will decompose

$$v = v_1 + v_2 + v_3 \quad \text{with } v_1 \in \mathfrak{M}_E^{E-}, v_2 \in \mathfrak{R}_{\Psi}^{E-}, v_3 \in \mathfrak{M}_{\Psi \setminus E}^-.$$

Our bounded domain in $\mathfrak{M}_{\Psi \setminus E}$ is

$$D_{\Psi \setminus E} = \xi^{-1}G_{\Psi \setminus E,0}(x_0).$$

The real form of \mathfrak{M}_E^{E-} is $\text{ad}(c_E) \{ \mathfrak{G}_0 \cap \text{ad}(c_E^{-1})\mathfrak{M}_E^{E-} \}$, and the positive definite inner product on that real form is induced by the positive definite hermitian inner product $\langle u, v \rangle = -\langle u, \sigma\tau v \rangle$ on \mathfrak{G} . The self dual cone $\mathfrak{C}_E \subset \text{ad}(c_E) \{ \mathfrak{G}_0 \cap \text{ad}(c_E^{-1})\mathfrak{M}_E^{E-} \}$ requires some notation. If $F \subset \Psi$ we have $K_F = K \cap G_{F,0}$ and the fixed point set K_F^E of $\text{ad}(c_E^4)$ on K_F ; now $\text{ad}(c_E^2)$ restricts to an involutive automorphism of \mathfrak{K}_F^E , and that defines an algebra

$$\mathfrak{K}_F^{E*} = \{ v \in \mathfrak{K}_F^E : \text{ad}(c_E^2)v = v \} + i\{ v \in \mathfrak{K}_F^E : \text{ad}(c_E^2)v = -v \}.$$

Its analytic group is denoted K_F^{E*} . Now the cone is given, in our notation, by

$$\mathfrak{C}_E = \text{ad}(K_E^{E*}) \cdot \xi^{-1}(x_{E,\phi}) \quad \text{and} \quad \xi^{-1}(x_{E,\phi}) = \sum_E e_{-\psi}.$$

If $w \in \mathfrak{M}_{\Psi \setminus E}$ then $\mu(w) : \mathfrak{R}_{\Psi}^{E-} \rightarrow \mathfrak{R}_{\Psi}^{E-}$ is defined by $\mu(w)v = \text{ad}(w) \cdot \text{ad}(c_E^2) \cdot \sigma\tau(v)$; it has operator norm < 1 if and only if $w \in D_{\Psi \setminus E}$. Thus if $w \in D_{\Psi \setminus E}$, $I + \mu(w)$ is invertible and we have $F_w : \mathfrak{R}_{\Psi}^{E-} \times \mathfrak{R}_{\Psi}^{E-} \rightarrow \mathfrak{M}_E^{E-}$ by

$$F_w(u, v) = - (i/2)[u, \text{ad}(c_E^2) \cdot \sigma\tau \cdot (I + \mu(w))^{-1}v].$$

The main result of [15, §7] says first that

(11A.1) $\mathfrak{C}_E = \{ v \in \mathfrak{M}^- : v_3 \in D_{\Psi \setminus E} \text{ and } \text{Im } v_1 - \text{Re } F_{v_1}(v_2, v_2) \in \mathfrak{C}_E \}$ is a Siegel domain of type III with base $D_{\Psi \setminus E}$, and second that

(11A.2) $c_E X_0 \subset \xi(\mathfrak{M}^-) \quad \text{and} \quad \xi^{-1} : c_E X_0 \cong \mathfrak{C}_E.$

In proving the results stated above, Wolf and Korányi use certain symmetry properties

$$\text{ad}(k)F_w(u, v) = F_{\text{ad}(k)w}(\text{ad}(k)u, \text{ad}(k)v) \quad \text{for } k \in K_{\Psi}^{E*},$$

$$F_w(u, (I + \mu(w))v) = F_0(u, v) \quad \text{and} \quad F_0(u, \mu(w)v) = F_0(v, \mu(w)u)$$

of the F_w and rely on the ξ^{-1} -equivariant actions of $\text{ad}(c_E)Z_{E,\phi,0}$ and $\text{ad}(c_E)N_{E,\phi,0}$ (cf.(10.10)) from the natural action on $c_E X_0$ to the (defined by equivariance) action on \mathfrak{M}^- . For if N^E denotes the common unipotent radical of those groups,

$$N^E = \{ \text{ad}(c_E)N_{E,\phi,0} \}^u = \{ \text{ad}(c_E)Z_{E,\phi,0} \}^u,$$

then [15, Theorem 6.8] N^E is 2-step nilpotent, its Lie algebra \mathfrak{N}^E has complexification

$$\mathfrak{N}^{EC} = \text{ad}(c_E)\mathfrak{N}_{E,\phi}^u = \text{ad}(c_E)Z_{E,\phi}^u = (\mathfrak{Q}_\Psi^{E+} + \mathfrak{R}_\Psi^{E-}) + \mathfrak{N}_E^{E-},$$

and $\text{ad}(c_E)Z_{E,\phi,0}$ and $\text{ad}(c_E)N_{E,\phi,0}$ have respective identity components

$$\{L_{E,\phi,0} \cdot K_E^{E*}\} \cdot N^E \quad \text{and} \quad \{G_{\Psi \setminus E,0} \cdot L_{E,\phi,0} \cdot K_E^{E*}\} \cdot N^E;$$

here we recall $\mathfrak{G}_{\Psi,0}^E = \mathfrak{G}_{\Psi \setminus E,0} \oplus \mathfrak{L}_{E,\phi,0} \oplus \mathfrak{G}_{E,0}^E$, $\mathfrak{L}_{E,\phi,0} \subset \mathfrak{K}$, so that

$$K_\Psi^{E*} = K_{\Psi \setminus E} \cdot L_{E,\phi,0} \cdot K_E^{E*}.$$

The action of K_Ψ^{E*} on \mathfrak{N}^- is induced by ad_G ; it preserves each of \mathfrak{N}_E^{E-} , \mathfrak{R}_Ψ^{E-} and $\mathfrak{N}_{\Psi \setminus E}$, the real form specified of \mathfrak{N}_E^{E-} and the cone \mathfrak{C}_E there, the bilinear maps F_w and the domain $D_{\Psi \setminus E}$, and thus also the Siegel domain \mathfrak{S}_E . The action of N^E on \mathfrak{N}^- is based on the unique representation of every $g \in N^E$ in the form $\exp(u + (I - \text{ad}(c_E^2)\sigma\tau)v)$ with $u \in \mathfrak{N}_E^{E-}$ and $v \in \mathfrak{R}_\Psi^{E-}$; if

$$f: \mathfrak{R}_\Psi^{E-} \times \mathfrak{N}_{\Psi \setminus E}^- \rightarrow \mathfrak{R}_\Psi^{E-} \quad \text{by} \quad (v, w) \rightarrow f_\bullet(v, w) = v + \mu(w)v,$$

then N^E acts on \mathfrak{N}^- by

$$g(e) = e + u + f_\bullet(v, e_3) + 2iF_{e_3}(e_2, f_\bullet(v, e_3)) + iF_{e_3}(f_\bullet(v, e_3), f_\bullet(v, e_3));$$

that action preserves the Siegel domain \mathfrak{S}_E . The action of $G_{\Psi \setminus E,0}$ on \mathfrak{N}^- or \mathfrak{S}_E is not made explicit in [15], but the action of $K_{\Psi \setminus E} \subset K_\Psi^{E*}$ is described above, and $G_{\Psi \setminus E,0}$ acts on $D_{\Psi \setminus E}$ in the natural fashion.

11A.3. PROPOSITION. *Let \mathfrak{S}_E be the Siegel domain (11A.1) with base $D_{\Psi \setminus E}$ that is holomorphically equivalent (by $c_E^{-1} \cdot \xi$) to X_0 . Define*

$$\pi: \mathfrak{S}_E \rightarrow D_{\Psi \setminus E} \quad \text{by} \quad \pi(v) = v_3.$$

1. $\pi: \mathfrak{S}_E \rightarrow D_{\Psi \setminus E}$ is a holomorphic fibre bundle.
2. The identity component $\{G_{\Psi \setminus E,0} \cdot L_{E,\phi,0} \cdot K_E^{E*}\} \cdot N^E$ of $\text{ad}(c_E)N_{E,\phi,0}$ acts transitively on \mathfrak{S}_E by holomorphic bundle automorphisms, with $G_{\Psi \setminus E,0}$ transitive on the base and the identity component $\{L_{E,\phi,0} \cdot K_E^{E*}\} \cdot N^E$ of $\text{ad}(c_E)Z_{E,\phi,0}$ trivial on the base and transitive on every fibre.
3. The fibre $\pi^{-1}(0)$ is the Siegel domain $\{v \in \mathfrak{N}_E^{E-} + \mathfrak{R}_\Psi^{E-} : \text{Im } v_1 - F_0(v_2, v_2) \in \mathfrak{C}_E\}$ of type II homogeneous under $\{L_{E,\phi,0} \cdot K_E^{E*}\} \cdot N^E$, and it is holomorphically equivalent to a hermitian symmetric space of noncompact type that has rank $|E|$ and tube part $G_{E,0}^E(x_0)$.

4. The bundle $\pi: \mathfrak{S}_E \rightarrow D_{\Psi \setminus E}$ is real-analytically trivial, and it is holomorphically trivial if and only if E is a union of sets $\Psi \cap \Delta^k$ where the Δ^k are the root systems of the simple ideals \mathfrak{G}^k of \mathfrak{G} .

PROOF. Let $e = e_1 + e_2 + e_3 \in \mathfrak{S}_E$ and represent $g \in N^E$ in the form $\exp(u + (I - \text{ad}(c_E^2)\sigma\tau)v)$ with $u \in \mathfrak{M}_E^{E-}$ and $v \in \mathfrak{R}_\Psi^{E-}$. Let $e' = g(e)$; then

$$e' = e + u + f_\nu(e_3) + 2iF_{e_3}(e_2, f_\nu(e_3)) + iF_{e_3}(f_\nu(e_3), f_\nu(e_3)),$$

so $e'_3 = e_3$, $e'_2 = e_2 + f_\nu(e_3)$ and

$$e'_1 = e_1 + u + 2iF_{e_3}(e_2, f_\nu(e_3)) + iF_{e_3}(f_\nu(e_3), f_\nu(e_3)).$$

Thus the action of N^E on \mathfrak{S}_E preserves every π -fibre. If $k \in L_{E, \phi, 0} \cdot K_E^{E*}$, then $k(e) = \text{ad}(k)e$, so $k(e)_3 = \text{ad}(k)e_3 = e_3$. Thus the action of $L_{E, \phi, 0} \cdot K_E^{E*}$, on \mathfrak{S}_E preserves every π -fibre. Define

$$v = -(I + \mu(e_3))^{-1}e_2 \quad \text{and} \quad g = \exp((I - \text{ad}(c_E^2)\sigma\tau)v) \in N^E.$$

Then $g(e) = e'_1 + 0 + e_3$. Now define

$$u = -\text{Re } e'_1 \quad \text{and} \quad g' = \exp(u) \in N^E.$$

Then $g'g(e) = e'_1 + 0 + e_3$ with ie'_1 real. In fact $g'g(e) \in \mathfrak{S}_E$ gives us $\text{Im } e'_1 - \text{Re } F_{e_3}(0, 0) = \text{Im } e'_1 \in \mathfrak{C}_E$. Thus there exists $k \in K_E^{E*}$ such that $\text{ad}(k) \cdot \text{Im } e'_1 = \xi^{-1}(x_{E, \phi})$. Now $kg'g \in K_E^{E*} \cdot N^E$ has the property that

$$(kg'g)(e) = i\xi^{-1}(x_{E, \phi}) + e_3.$$

We have proved that the π -fibres are just the orbits of $\{L_{E, \phi, 0} \cdot K_E^{E*}\} \cdot N^E$ on \mathfrak{S}_E .

Let $g \in G_{\Psi \setminus E, 0}$. Then g normalizes $\{L_{E, \phi, 0} \cdot K_E^{E*}\} \cdot N^E$ in $\{G_{\Psi \setminus E, 0} \cdot L_{E, \phi, 0} \cdot K_E^{E*}\} \cdot N^E$, hence permutes its orbits on \mathfrak{S}_E . Now g is a π -fibre map of \mathfrak{S}_E . Thus (2) is proved subject to (1) being demonstrated.

Let Q denote the identity component of $\text{ad}(c_E)N_{E, \phi, 0}$, Y its isotropy subgroup at $i\xi^{-1}(x_{E, \phi}) \in \mathfrak{S}_E$, and Z the subgroup preserving $\pi^{-1}(0)$. Then $Y \subset Z = \{K_{\Psi \setminus E} \cdot L_{E, \phi, 0} \cdot K_E^{E*}\} \cdot N^E = K_\Psi^{E*} \cdot N^E$ and π is projection $Q/Y \rightarrow Q/Z$ by $qY \rightarrow qZ$. Thus π is a differentiable fibre bundle with structure group Z . But π visibly is a holomorphic map, and the fibre over $g(0)$, $g \in G_{\Psi \setminus E, 0}$, is the complex submanifold

$$g \cdot \pi^{-1}(0) = g \cdot \{v \in \mathfrak{M}_E^{E-} + \mathfrak{R}_\Psi^{E-} : \text{Im } v_1 - F_0(v_2, v_2) \in \mathfrak{C}_E\}$$

of \mathfrak{S}_E . As Z acts by holomorphic transformations of $\pi^{-1}(0)$, it follows that $\pi: \mathfrak{S}_E \rightarrow D_{\Psi \setminus E}$ is a holomorphic fibre bundle. That completes the proof of (1) and (2).

Every π -fibre is of the form $g \cdot \pi^{-1}(0)$, $g \in G_{\Psi \setminus E, 0}$, and $G_{\Psi \setminus E, 0}$ normalizes the transitive group $\{L_{E, \phi, 0} \cdot K_E^{E*}\} \cdot N^E$ of transformations of the fibre; thus an arbitrary π -fibre has the same properties as $\pi^{-1}(0)$ as complex homogeneous space of $\{L_{E, \phi, 0} \cdot K_E^{E*}\} \cdot N^E$. Now we observe

$$\pi^{-1}(0) = \{v \in \mathfrak{M}_E^{E-} + \mathfrak{R}_\Psi^{E-} : \text{Im } v_1 - F_0(v_2, v_2) \in \mathfrak{C}_E\}$$

where $F_0: \mathfrak{R}_\Psi^{E-} \times \mathfrak{R}_\Psi^{E-} \rightarrow \mathfrak{M}_E^{E-}$ is the positive definite (relative to \mathfrak{C}_E) hermitian map given by

$$F_0(u, v) = - (i/2)[u, \text{ad}(c_E^2)\sigma\tau v].$$

Thus $\pi^{-1}(0)$, hence every fibre, is a homogeneous Siegel domain of type II. Comparison with [2, §6] suggests that it should be symmetric, and we prove that symmetry as follows.

Observe that $\{v \in \pi^{-1}(0) : v_2 = 0\}$ is the tube domain over the self dual cone \mathfrak{C}_E ; in fact that is the realization of $G_{E, 0}^E(x_0)$ as a Siegel domain of type I. Thus there is an analytic automorphism δ of order 2 on that tube domain with $i\xi^{-1}(x_{E, \phi})$ as the only fixed point. Given $e \in \pi^{-1}(0)$, define

$$v = -e_2 \quad \text{and} \quad g_e = \exp((I - \text{ad}(c_E^2)\sigma\tau)v) \in N^E.$$

Then $e' = g_e(e)$ satisfies $e'_2 = 0$, $e'_1 = e_1 + f_{-e_2}(0)$ and

$$e'_1 = e_1 + 2iF_0(e_2, f_{-e_2}(0)) + iF_0(f_{-e_2}(0), f_{-e_2}(0)).$$

As $f_{-e_2}(0) = -e_2 + \mu(0)(-e_2) = -e_2$, now $e'_1 = e_1 + 2iF_0(e_2, -e_2) + iF_0(-e_2, -e_2) = e_1 - iF_0(e_2, e_2)$, $e'_2 = 0$ and $e'_3 = 0$. Thus

$$g_e(e) = e_1 - iF_0(e_2, e_2) \in \{v \in \pi^{-1}(0) : v_2 = 0\}.$$

Now define $e^* = g_e \delta g_e(e)$. It satisfies $e_1^* = \delta g_e(e) + 2iF_0(0, f_{-e_2}(0)) + iF_0(f_{-e_2}(0), f_{-e_2}(0)) = \delta g_e(e) + iF_0(e_2, e_2)$ and $e_2^* = f_{-e_2}(0) = -e_2$. Thus

$$e^* = g_e \delta g_e(e) = \delta(e_1 - iF_0(e_2, e_2)) + iF_0(e_2, e_2) - e_2.$$

Now

$$\begin{aligned} (e^*)^* &= \delta(\delta(e_1 - iF_0(e_2, e_2)) + iF_0(e_2, e_2) - iF_0(e_2, e_2)) + iF_0(e_2, e_2) + e_2 \\ &= \delta^2(e_1 - iF_0(e_2, e_2)) + iF_0(e_2, e_2) + e_2 \\ &= e_1 - iF_0(e_2, e_2) + iF_0(e_2, e_2) + e_2 = e_1 + e_2 = e. \end{aligned}$$

Thus $e \rightarrow e^*$ is an analytic automorphism of $\pi^{-1}(0)$ with square 1. If $e = e^*$ then

$$e_1 = \delta(e_1 - iF_0(e_2, e_2)) + iF_0(e_2, e_2) \quad \text{and} \quad e_2 = -e_2.$$

Thus $e_2 = 0$ and $e_1 = \delta(e_1)$. The latter says $e_1 = i\xi^{-1}(x_{E,\phi})$. Thus $e = e^*$ implies $e = i\xi^{-1}(x_{E,\phi})$. Now $e \rightarrow e^*$ is an analytic automorphism of order 2 on $\pi^{-1}(0)$ with $i\xi^{-1}(x_{E,\phi})$ as unique fixed point. As $\pi^{-1}(0)$ is a homogeneous complex manifold holomorphically equivalent to a bounded domain, it follows that $\pi^{-1}(0)$ is a hermitian symmetric space of noncompact type. The tube part of that symmetric space is $\{v \in \pi^{-1}(0) : v_2 = 0\}$, tube domain over \mathbb{C}_E , holomorphically equivalent to $G_{E,0}^{\mathbb{R}}(x_0)$; in particular $\pi^{-1}(0)$ has rank $|E|$. Statement (3) is proved.

$G_{\Psi \setminus E, 0}$ has Iwasawa decomposition $K_{\Psi \setminus E} \cdot A_{\Psi \setminus E} \cdot U_{\Psi \setminus E}$ and the solvable group $B = A_{\Psi \setminus E} \cdot U_{\Psi \setminus E}$ is simply transitive on $D_{\Psi \setminus E}$. Now define

$$\lambda: \pi^{-1}(0) \times D_{\Psi \setminus E} \rightarrow \mathbb{S}_E \quad \text{by} \quad \lambda(t, b(0)) = b(t).$$

Then λ is a real analytic trivialization of $\pi: \mathbb{S}_E \rightarrow D_{\Psi \setminus E}$.

Let \mathfrak{g}^k denote the simple ideals of \mathfrak{g} , Δ^k the root system of \mathfrak{g}^k . If E is a union of sets $\Psi \cap \Delta^k$, then $G_0(x_0) = G_{\Psi \setminus E, 0}(x_0) \times G_{E, 0}(x_0)$ and π projects to the first factor. Conversely suppose π holomorphically trivial. Let D'_E denote the bounded symmetric domain in $\mathfrak{N}_E^{\mathbb{R}-} + \mathfrak{A}_{\Psi}^{\mathbb{R}-}$ holomorphically equivalent to $\pi^{-1}(0)$. Then there is a holomorphic equivalence of $G_0(x_0) \cong \mathbb{S}_E$ with $D_{\Psi \setminus E} \times D'_E$. As G_0 is the largest connected group of holomorphic automorphisms of $G_0(x_0)$, as $G_{\Psi \setminus E, 0}$ is the largest connected group for $D_{\Psi \setminus E}$, and as the Bergman metric on $D_{\Psi \setminus E} \times D'_E$ is the direct sum of the Bergman metrics of $D_{\Psi \setminus E}$ and of D'_E , it follows that $G_0 = G_{\Psi \setminus E, 0} \times G_{E, 0}$ where $G_{E, 0}$ is the largest connected group of holomorphic automorphisms of D'_E . Thus $\mathfrak{g}_{\Psi \setminus E}$ is a sum of simple ideals of \mathfrak{g} , so $\Psi \setminus E$ is a union of sets $\Psi \cap \Delta^k$, and we conclude that E is a union of sets $\Psi \cap \Delta^k$. Statement (4) is proved, and that completes the proof of Proposition 11A.3. Q.E.D.

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