MINIMAL VARIETIES[1]

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ABSTRACT. This is a survey article, reporting on recent results in the theory of minimal varieties in euclidean space, and including a number of theorems on minimal submanifolds of spheres.

Introduction. It was exactly 100 years ago, in 1868, that Beltrami presented the first general survey of the theory of minimal surfaces [4]. This survey has been referred to by Blaschke [5, p. 118] as representing the "stormy youth" of the subject, in contrast to its "tired old age" in the nineteen thirties. Although I would take issue with both of Blaschke's characterizations, I think it incontestable that the last ten years have seen a "stormy rebirth" of the theory of minimal surfaces. One of the most striking developments, although certainly not the only one, has been the creation of a theory of higher-dimensional minimal varieties. Since several recent surveys (Nitsche [35], Osserman [36(c)]) have been devoted to two-dimensional minimal surfaces, the goal here will be to concentrate on the higher-dimensional case, and restrict the discussion of two-dimensional surfaces to some of the most recent results. Furthermore, we shall discuss only minimal varieties in euclidean spaces, except for §6, which deals also with minimal submanifolds of spheres. Since much recent work has been devoted to minimal submanifolds of spheres, and some to arbitrary Riemannian manifolds, we have included these topics in the list of references. Note in particular the set of lecture notes by Chern [10(b)].

There are many points of view from which minimal varieties may be studied. The emphasis here will be on their differential-geometric properties, and on the associated differential equations. For detailed discussions relative to the calculus of variations and to measure the-

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ory, see the recent books of Morrey [33] and Federer [19]. These books present, in particular, extensive treatments of Plateau's problem, which we shall not discuss here. In this connection, see also the book of Almgren [2(b)], and the recent papers of Allard [1], Almgren [2(c)], and Hildebrandt [25].[2]

1. Basic notation and definitions; variational formula. We start with a brief review of some basic facts concerning local properties of m-dimensional manifolds in R^n. For further details, see Osserman [36(c)] for the case $m=2$, and Eisenhart [18] for arbitrary m.

Let $u=(u_1, \cdots, u_m)$ and $x=(x_1, \cdots, x_n)$ denote points in R^m and R^n respectively. Let D be a domain in R^m and let

$$x(u): \ D \to R^n$$

be a differentiable map. We do not specify the order of differentiability, since it is generally straightforward to verify what order is needed in a particular case. For most of our considerations, C^2 or C^3 is sufficient.

We introduce the standard notation of classical differential geometry:

$$\begin{aligned} g_{ij} &= \frac{\partial x}{\partial u_i}\cdot\frac{\partial x}{\partial u_j} = \sum_{k=1}^{n} \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j}, \qquad i, j = 1, \cdots, m, \\ g &= \det(g_{ij}), \qquad (g^{ij}) = (g_{ij})^{-1}. \end{aligned} \tag{1.1}$$

Thus, the g^{ij} are the elements of the inverse matrix to (g_{ij}), and the existence of this inverse is one of our basic assumptions. Namely, the map $x(u)$ is called *regular* if the following equivalent properties hold.

(i) $g\neq 0$,

(ii) $g>0$,

(iii) the jacobian matrix $(\partial x_i/\partial u_j)$ has rank m,

(iv) the tangent vectors $\partial x/\partial u_1, \cdots, \partial x/\partial u_m$ are linearly independent.

The equivalence of (i), (ii), (iii) follows immediately from the identity

$$\det(g_{ij}) = \sum_{1\le i_1<\cdots<i_m\le n} \left[\frac{\partial(x_{i_1}, \cdots, x_{i_m})}{\partial(u_1, \cdots, u_m)}\right]^2.$$

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[2] *Added in proof* (*September 22, 1969*). Since this paper was written two important contributions have been made to the classical solution of Plateau's problem, showing interior and boundary regularity respectively. For this, see Osserman [56] and Nitsche [55(b)]. Also Hildebrandt [51], Heinz and Toni [49], and Kinderlehrer [53].

Let $x(u)$ be a regular map of the above form, and assume further that $x(u)$ is a one-one map of the domain D onto a set M in R^n. Given any point $p \in M$, let $p=x(a)$ for $a \in D$. Then condition (iv) guarantees that the vectors $(\partial x/\partial u_1)(a), \cdots, (\partial x/\partial u_m)(a)$ span an m-dimensional space called the *tangent space* to M at p and denoted by $T_p(M)$. The orthogonal complement to $T_p(M)$ in R^n is an $(n-m)$-dimensional space called the *normal space* to M at p and denoted by $N_p(M)$.

Let $F=D' \cup \partial D'$ be a compact subset of D, where D' is a domain with smooth boundary $\partial D'$. The image of F on M has *m-dimensional volume* equal to

$$\int_F \sqrt{g}\, du_1 \cdots du_m.$$

We wish to consider the variation in volume associated with a variation in the mapping $x(u)$. Let

$$x(u;t)\colon\ D \times I \to R^n$$

define a one-parameter family of mappings differentiable in $D \times I$, where I is some interval about 0 on the real line, such that $x(u;0)$ coincides with our original map $x(u)$. For each t we have corresponding quantities $g_{ij}(t)$ and $g(t)$. Since $g(t)$ depends continuously on t, the regularity of $x(u)$ implies the regularity of $x(u;t)$ on some domain containing F for all sufficiently small t. The volume of the image of F under $x(u;t)$ is

$$V(t) = \int_F \sqrt{g(t)}\, du_1 \cdots du_m. \tag{1.2}$$

We would like a formula for the *first variation* $V'(t)$ at $t=0$. For this purpose we consider at each point $p=x(a)$ of M the variation vector

$$E = \frac{d}{dt} x(a;t)\Big|_{t=0}.$$

The vector E is simply the tangent vector at p to the curve described by the image of a fixed point $a \in D$ as t varies. We decompose it into its tangent and normal components:

$$E = E^T + E^N, \quad \text{where} \quad E^T \in T_p(M),\ E^N \in N_p(M).$$

If $E^T=0$ at each point, the variation is called a *normal variation.* In that case, the variational formula is particularly simple. The effect of the variation is determined by the variational vector field

$E=E^N$, the precise expression being

$$V'(0) = -\int_F E\cdot H\sqrt{g}\,du_1 \cdots du_m, \tag{1.3}$$

where H is a vector field determined by the manifold M. The vector H at each point p of M is called the *mean curvature vector* of M at p. We shall derive various expressions for it below. First let us note that it is a normal vector: $H\in N_p(M)$, and that the variational formula (1.3) provides important information about the significance of H.

To begin with, we observe that the vector field E may be prescribed arbitrarily and a corresponding variation may be formed in many ways. The simplest would be

$$x(u;t) = x(u) + tE. \tag{1.4}$$

In particular, if we choose $E=H$, then we find

$$V'(0) = -\int_F |H|^2\sqrt{g}\,du_1 \cdots du_m, \tag{1.5}$$

so that $V'(0)<0$ unless $H\equiv 0$. In fact, if $H\equiv 0$, then equation (1.3) shows that the first variation of volume is zero for *every* normal variation. On the other hand, if at some point $p\in M$, $H\neq 0$, then we may choose a variation vector field of the form $E=\lambda H$ where $\lambda(p)>0$, $\lambda\geqq 0$ everywhere, and $\lambda\equiv 0$ outside an arbitrarily small neighborhood N of p. Then (1.4) gives a variation which leaves M fixed outside N and which strictly decreases volume.

In view of these facts, M is called *minimal* if $H\equiv 0$.

Equation (1.5) provides a valuable intuitive interpretation of the mean curvature vector H. Namely, it may be pictured as pointing toward the "inside" of M, in the sense that if M is deformed by moving each point in the direction of the mean curvature vector at that point, then the volume of M will initially decrease.

In the case $m=1$, M is curve $x(u)$ in R^n, and regularity means that $x'(u)\neq 0$ which implies that M may be parameterized with respect to arc length s. There is at each point a well-defined *unit tangent vector*

$$T = \frac{dx}{ds} = \frac{dx}{du}\Big/\left|\frac{dx}{du}\right|$$

and a *curvature vector*

$$dT/ds = d^2x/ds^2.$$

In this case, F is an interval $[\alpha, \beta]$ on the real line, and instead of volume we have arc length

$$L = \int_\alpha^\beta |x'(u)|\, du.$$

Corresponding to a variation with variation vector field E, we have the formula for variation in arc length:

$$L'(0) = -\int_\alpha^\beta E \cdot H\, ds$$

where H is simply the curvature vector d^2x/ds^2. Thus a one-dimensional minimal variety in R^n is a straight line. Furthermore, the interpretation of H as indicating the "inside" of M is clear here, since the curvature vector, defined as the derivative of the unit tangent T, is easily seen as directed toward what we would describe as the "inside" of a curve. (In particular, at any point p where the curvature vector H is nonzero, if we consider the hyperplane through p perpendicular to H, then the curve lies locally on that side of the hyperplane indicated by H.)

For arbitrary m, the mean curvature vector H at a point p of M may be described in terms of the curvature vectors of all regular curves through p lying on M. Namely, if $x(s)$ is such a curve, where s is the parameter of arc length, then $x(s)$ is the image of a curve $u(s)$ lying in D, and

$$\frac{dx}{ds} = \sum_{i=1}^{m} u_i'(s) \frac{\partial x}{\partial u_i}, \tag{1.6}$$

$$\frac{d^2x}{ds^2} = \sum_{i,j=1}^{m} u_i'(s)u_j'(s) \frac{\partial^2 x}{\partial u_i \partial u_j} + \sum_{i=1}^{m} u_i''(s) \frac{\partial x}{\partial u_i}. \tag{1.7}$$

Since $\partial x/\partial u_i$ is a tangent vector to M, the component of the curvature vector normal to M is given by

$$\left(\frac{d^2x}{ds^2}\right)^N = \sum u_i'(s)u_j'(s) \left(\frac{\partial^2 x}{\partial u_i \partial u_j}\right)^N = \sum u_i'(s)u_j'(s) B_{ij} \tag{1.8}$$

where we use the notation

$$B_{ij} = \left(\frac{\partial^2 x}{\partial u_i \partial u_j}\right)^N. \tag{1.9}$$

Since the unit tangent $T=dx/ds$ is determined by and determines the quantities $u_i'(s)$ by (1.6), it follows from (1.8) that the normal component of the curvature vector is the same for all curves on M having a given tangent direction T at p. It is called the ***normal curvature vector*** of M at p in the direction T, and we denote it by $k(T)$. Thus, by (1.8) and (1.9),

$$k(T) = \sum u_i'(s)u_j'(s)B_{ij} \tag{1.10}$$

where

$$T = \sum u_i'(s)\partial x/\partial u_i.$$

Another interpretation of $k(T)$ is the following. Let L be the $(n-m+1)$-dimensional affine subspace of R^n through p, generated by T and $N_p(M)$. By the implicit function theorem, $L\cap M$ defines a regular curve near p called the *normal section* of M in the direction T. The curvature vector of this curve lies in $N_p(M)$ and hence coincides with the normal curvature vector $k(T)$.

Let $T_1, \cdots, T_m$ be an orthonormal basis of $T_p(M)$. Then the mean curvature vector H of M at p is given by

$$H = k(T_1) + \cdots + k(T_m),$$

the sum of the normal curvatures on the right being independent of the choice of basis.

Analytically, we introduce at each point p of M the *first fundamental form*, which is a map $T_p(M)\to R$ defined by

$$\sum \xi_i \frac{\partial x}{\partial u_i} \mapsto \left| \sum \xi_i \frac{\partial x}{\partial u_i}\right|^2 = \sum g_{ij}\xi_i\xi_j,$$

and the *second fundamental form*, a map

$$T_p(M) \to N_p(M)$$

defined by

$$\sum \xi_i \frac{\partial x}{\partial u_i} \mapsto \sum \xi_i\xi_j B_{ij}.$$

The values of the second fundamental form, under the constraint that the first fundamental form be equal to one, represent the totality of normal curvature vectors to M at p. It follows that the mean curvature vector is given by

$$H = \sum_{i,j=1}^{n} g^{ij} B_{ij} = \left(\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2 x}{\partial u_i \partial u_j} \right)^N. \tag{1.11}$$

The equation (1.3) for the first variation may be derived by differentiating (1.2) under the integral sign and using (1.11) together with (1.1) and the identity

$$\frac{dg}{dt} = g \sum_{i,j} g^{ij} \frac{dg_{ij}}{dt} \tag{1.12}$$

which holds for the derivative of the determinant of any nonsingular matrix.

We return briefly to the variational formula in the case of an arbitrary (not necessarily normal) variation. It turns out that the terms involving the tangential part E^T of the variational vector field may be reduced to an integral over the boundary of F. If, in particular, the boundary is held fixed, the formula becomes

$$V'(0) = -\int_F E^N \cdot H \sqrt{g}\, du_1 \cdots du_n.$$

Thus, the condition $H \equiv 0$ means that the volume is stationary for all variations which keep the boundary fixed.

2. **The Laplace-Beltrami operator; nonparametric representation.** Given a regular one-to-one map

$$x(u): D \to M \subset R^n,$$

we define the *Laplace-Beltrami operator* Δ on M for every C^2 function $\phi: M \to R$ by

$$\Delta\phi = \frac{1}{\sqrt{g}} \sum_i \frac{\partial}{\partial u_i} \left(\sqrt{g} \sum_j g^{ij} \frac{\partial \phi}{\partial u_j} \right). \tag{2.1}$$

The function ϕ is *harmonic* if $\Delta\phi \equiv 0$.

If we apply the operator Δ to each coordinate function x_k we obtain a vector

$$\Delta x = (\Delta x_1, \cdots, \Delta x_n).$$

LEMMA 2.1. *$\Delta x \in N_p(M)$ at each point $p = x(u)$.*

PROOF. For each k, one finds $\Delta x \cdot \partial x / \partial u_k = 0$ by a computation, using the identity (1.12) with $t = u_k$.

THEOREM 2.1. *For an arbitrary regular map* $x(u)$,

$$\Delta x = H. \tag{2.2}$$

PROOF. Using (2.1),

$$\Delta x = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial u_i} (\sqrt{g}\, g^{ij}) \frac{\partial x}{\partial u_j} + \sum g^{ij} \frac{\partial^2 x}{\partial u_i \partial u_j} \cdot \tag{2.3}$$

Since the first term on the right is tangent to M, we have by (1.11)

$$H = \left(\sum g^{ij} \frac{\partial^2 x}{\partial u_i \partial u_j} \right)^N = (\Delta x)^N = \Delta x.$$

COROLLARY. *M is minimal if and only if each coordinate function x_k is harmonic on M.*

We turn next to a special set of parameters on M. By virtue of property (iii) in the definition of regularity, together with the implicit function theorem, it follows that each point of M has a neighborhood which may be represented by solving for $n-m$ of the coordinates x_k in terms of the other m coordinates. By relabeling the coordinates we may assume that this neighborhood on M may be represented in the form

$$x_k = f_k(x_1, \cdots, x_m), \qquad k = m+1, \cdots, n. \tag{2.4}$$

This is called a *nonparametric* representation of the neighborhood on M. All previous formulas may be applied, substituting $u_k = x_k$, $k = 1, \cdots, m$.

THEOREM 2.2. *Let a manifold M be given in the nonparametric form* (2.4). *Then the following statements are equivalent.*

(a) *M is minimal*;

(b) *the functions f_k satisfy the equations*

$$\sum_{i,j=1}^{m} g^{ij} \frac{\partial^2 f_k}{\partial x_i \partial x_j} = 0, \qquad k = m+1, \cdots, n; \tag{2.5}$$

(c) *the functions f_k satisfy equations* (2.5) *together with*

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i} (\sqrt{g}\, g^{ij}) = 0, \qquad j = 1, \cdots, m. \tag{2.6}$$

REMARK. The quantities g_{ij}, g^{ij}, and g in equations (2.5) and (2.6) are of course computed with respect to the parameters $x_1, \cdots, x_m$. Specifically, the g_{ij} are quadratic polynomials in the partial derivatives $\partial f_k / \partial x_i$.

PROOF. Equation (2.3) yields

$$\Delta x_k = \frac{1}{\sqrt{g}} \sum_{i=1}^{m} \frac{\partial}{\partial x_i} (\sqrt{g}\, g^{ik}), \qquad k = 1, \cdots, m.$$

Thus $H=0 \Rightarrow \Delta x_k = 0 \Rightarrow$ equations (2.6) hold. Substituting back in (2.1) shows that if M is minimal, then

$$(2.7) \qquad \Delta\phi = \sum_{i,j=1}^{m} g^{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j}$$

for any function ϕ. In particular, using the coordinate functions $\phi = x_k = f_k(x_1, \cdots, x_m)$, $k = m+1, \cdots, n$ yields equations (2.5) whenever $H=0$. Thus (a)$\Rightarrow$(c). Clearly (c)$\Rightarrow$(b). Finally, to show (b)$\Rightarrow$(a), note that the equations

$$\sum_{i,j=1}^{m} g^{ij} \frac{\partial^2 x_k}{\partial x_i \partial x_j} = 0, \qquad k = 1, \cdots, m$$

hold trivially for any nonparametric representation, since the second derivatives $\partial^2 x_k / \partial x_i \partial x_j$ all vanish. Thus, if equations (2.5) hold, it follows that the second term on the right-hand side of (2.3) must vanish. But the first term is a linear combination of tangent vectors. Thus $\Delta x \in T_p(M)$, and it follows from Lemma 2.1 that $\Delta x = 0$. By Theorem 2.1, M is minimal.

The main point of Theorem 2.2 is that it shows that the local study of minimal varieties in Euclidean space is equivalent to the study of the elliptic system of partial differential equations (2.5). Historically, this equivalence has been used both to derive properties of minimal surfaces by applying results on differential equations and to obtain properties of solutions by using geometric methods.

A side benefit of Theorem 2.2 is that we have found an additional set of equations (2.6) which must be satisfied by any solution of (2.5). These equations seem to have been first noticed in the paper [**36(b)**], where they are used in the case $m=2$ to generalize to arbitrary n various results known earlier for $n=3$ (see §5 below). A different form of the same equations were first derived for $m=2$ in [**36(c)**].

The system (2.5) reduces to a single equation when $n=m+1$. It is this case which has yielded the most significant body of results to date. We study it in the following section.

3. **Nonparametric minimal hypersurfaces.** A hypersurface M of R^n is given in nonparametric form by a single function

$$x_n = f(x_1, \cdots, x_m), \qquad n = m+1. \tag{3.1}$$

With respect to the parameters $x_1, \cdots, x_m$, we have

$$g_{ij} = \delta_{ij} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}. \tag{3.2}$$

We find an expression for the inverse matrix (g^{ij}), following Flanders [20]. Denote the gradient vector of f by

$$p = (p_1, \cdots, p_m), \qquad p_k = \partial f/\partial x_k. \tag{3.3}$$

Then, according to (3.2), the matrix (g_{ij}) may be written as

$$(g_{ij}) = I + p^T p \tag{3.4}$$

where I is the identity matrix. Let c be a constant to be determined. Then

$$(I - cp^Tp)(I + p^Tp) = I + (1 - c - c|p|^2)p^Tp,$$

using $pp^T = |p|^2$. We make the right side reduce to the identity by choosing $c = 1/(1+|p|^2)$. Thus

$$g^{ij} = I - \frac{1}{1+|p|^2} p_i p_j. \tag{3.5}$$

Since the matrix $p^Tp = (p_ip_j)$ obviously has rank at most equal to 1 and eigenvector p^T with corresponding eigenvalue $|p|^2$, its eigenvalues are 0 with multiplicity $m-1$ and $|p|^2$. By (3.4), (g_{ij}) has eigenvalues 1 with multiplicity $m-1$ and $1+|p|^2$. Thus

$$g = \det(g_{ij}) = 1 + |p|^2, \tag{3.6}$$

and (3.5) becomes

$$g^{ij} = \delta_{ij} - \frac{1}{g}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}. \tag{3.7}$$

The condition (2.5) that M be minimal reduces to

$$\sum_{i=1}^{m} \frac{\partial^2 f}{\partial x_i^2} - \frac{1}{g}\sum_{i,j=1}^{m} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\frac{\partial^2 f}{\partial x_i \partial x_j} = 0. \tag{3.8}$$

This is easily seen to be equivalent to

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left(\frac{1}{\sqrt{g}} \frac{\partial f}{\partial x_i}\right) = 0, \qquad g = 1 + \sum_{k=1}^{m}\left(\frac{\partial f}{\partial x_k}\right)^2. \tag{3.9}$$

Equation (3.9) is perhaps the neatest form of the minimal hypersurface equation.

For hypersurfaces, the normal space $N_p(M)$ is one-dimensional at each point, and consists of scalar multiples of a unit normal vector ν. It is usual to choose ν so that the set of vectors

$$\partial x/\partial u_1, \cdots, \partial x/\partial u_m, \nu$$

has positive orientation. In the nonparametric case (3.1), it is immediately verified that the vector

$$\nu = \frac{1}{\sqrt{g}}\left(-\frac{\partial f}{\partial x_1}, \cdots, -\frac{\partial f}{\partial x_m}, 1\right) \tag{3.10}$$

is such a unit normal. Then

$$B_{ij} = b_{ij}\nu, \qquad H = h\nu \tag{3.11}$$

define the scalar second fundamental form and mean curvature respectively. Further,

$$\frac{\partial^2 x}{\partial x_i \partial x_j} = \left(0, \cdots, 0, \frac{\partial^2 f}{\partial x_i \partial x_j}\right)$$

implies

$$b_{ij} = B_{ij}\cdot\nu = \left(\frac{\partial^2 x}{\partial x_i \partial x_j}\right)^N \cdot \nu = \frac{\partial^2 x}{\partial x_i \partial x_j}\cdot \nu = \frac{1}{\sqrt{g}} \frac{\partial^2 f}{\partial x_i \partial x_j} \tag{3.12}$$

and

$$h = H\cdot\nu = \sum g^{ij} b_{ij} = \frac{1}{\sqrt{g}} \sum g^{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} = \sum \frac{\partial}{\partial x_i}\left(\frac{1}{\sqrt{g}} \frac{\partial f}{\partial x_i}\right). \tag{3.13}$$

Thus the scalar second fundamental form equals up to a scalar factor, the Hessian matrix of f, and the scalar mean curvature is just the left-hand side of equation (3.9).

4. **Properties of solutions of the minimal hypersurface equation.** In this section we list some of the most important theorems which have been proved about solutions of the minimal hypersurface equation (3.9). We make no attempt to give the proofs, but refer to the original papers.

We begin with a result of a technical nature, due to Bombieri, de Giorgi, and Miranda [8].

THEOREM 4.1. *Let f be a positive solution of* (3.9) *in a ball of radius ρ about the origin. There exist constants c_1, c_2, independent of f, such that*

$$|\nabla f(0)| \leqq c_1 \exp(c_2 f(0)/\rho).$$

An inequality of this kind was first obtained in the case $m=2$ by Finn. It has been successively sharpened by various authors until recently when Serrin [**43(a)**] found the precise values of the constants involved. In terms of the quantity $W^2=1+|\nabla f|^2$, he showed that for $m=2$,

$$W(0) \leqq \exp((\pi/2)f(0)/\rho).$$

By an *entire* solution of (3.9) we mean a solution defined on all of R^m.

THEOREM 4.2. *Every entire solution of* (3.9) *bounded on one side is constant.*

The proof of this uses Theorem 4.1 to show that ∇f is uniformly bounded (by c_1, since ρ may be chosen arbitrarily large), and then a result of Moser [**34**] which asserts that such a solution must be constant.

THEOREM 4.3. *For $n \leqq 8$, every entire solution of* (3.9) *is linear.*

The case $n=3$ of this theorem is a classical result of Bernstein. A new proof of Bernstein's theorem given by Fleming [**21**] became the basis for proving Theorem 4.3 in higher dimensions. This was done for $n=4$ by de Giorgi [**13**], $n=5$ by Almgren [**2(a)**], and $n=6$, 7, 8 by Simons [**44**]. A startling development was the subsequent discovery of Bombieri, de Giorgi, and Giusti [**7**].

THEOREM 4.4. *For $n>8$, there exist nonlinear entire solutions of* (3.9).

This result must surely rank as one of the most surprising ones in all of differential equations. A brief discussion of the work leading up to Theorems 4.3 and 4.4 is given in §6 below.

We consider next the question of singularities of solutions. Here we have a result of de Giorgi and Stampacchia [**14**].

THEOREM 4.5. *Let K be a compact set in a domain D lying in R^m. Let f be a solution of* (3.9) *in $D-K$. If the $(m-1)$-dimensional Hausdorff measure of K is zero, then f may be extended to a solution of* (3.9) *in all of D.*

This result has since been extended so that the set K need not be a compact subset of D. Thus, if D is the unit ball in R^3, a diameter would be a removable set.

The theory of the minimal surface equation, which is a nonuniformly elliptic equation, differs from the theory of uniformly elliptic equations in various ways. One example is the fact that in Theorems 4.3 and 4.5, one need not assume that the solution is bounded (or bounded on one side) as in the corresponding theorems for solutions of uniformly elliptic equations. Another example is the way the geometry of the boundary enters when studying the Dirichlet problem. The result here is the following.

THEOREM 4.6. *Let D be a bounded domain in R^m with C^2-boundary B. Necessary and sufficient that there exist a solution of* (3.9) *taking on arbitrarily prescribed continuous boundary values is that the mean curvature vector of B in R^m should be directed toward the interior at each point of B.*

REMARKS. The mean curvature vector is allowed to vanish. The fact is that if at some point the mean curvature vector is directed toward the exterior of B, then it is possible to find (arbitrarily smooth) boundary values for which no solution exists. The surprising part is that convexity, which is needed in the theory of uniformly elliptic equations, plays no role here. In the case $m=2$, however, the mean curvature vector is simply the curvature vector of the boundary curve, and the condition of Theorem 4.6 reduces to convexity of D. In higher dimensions, D need not even be simply connected. For example, when $m=3$, if B is a torus with suitably chosen radii, its mean curvature vector will point toward the interior at each point.

Theorem 4.6 was originally proved by Jenkins and Serrin [27] under the additional hypothesis that the boundary values be C^2. The subsequent discovery of the estimate in Theorem 4.1 made possible the extension to arbitrary continuous boundary values.

It has often been the case that a theorem first proved for minimal surfaces has pointed the way toward results of much greater generality. Theorem 4.6 has provided the impetus for deriving a general theory of boundary value problems which includes a number of previously isolated results. This theory has been developed by Serrin [43(b)]. He considers elliptic equations of the form

$$\sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} = b,$$

where the coefficients a_{ij} and b may depend on x, f, and ∇f. Certain expressions involving the coefficients lead to a classification of these equations into categories which display remarkably different behavior relative to the Dirichlet problem.

We note finally that there have been many papers dealing with solutions of equation (3.9) in addition to those referred to above. Some of them have been of a nature preliminary to the above theorems, and others have provided further information about solutions. Examples are Bombieri [6], Gilbarg [23], Miranda [32(a)–(f)], Stampacchia [45].

5. **Two-dimensional surfaces.** The main feature which distinguishes the theory of m-dimensional minimal varieties with $m>2$ from the case $m=2$ is the applicability of complex-variable methods to the latter. In this section we discuss some of the most recent results for two-dimensional surfaces in R^n.

A. *Intrinsic characterizations.* Let a positive definite symmetric matrix (g_{ij}) be given at each point of a plane domain D. We may ask if there exists a map $x(u)$: $D\rightarrow R^3$ which defines a minimal surface and whose first fundamental form corresponds to the given matrix (g_{ij}) at each point; i.e., the given g_{ij} are related to the map $x(u)$ by relation (1.1).

This question was first considered by Ricci [40]. His answer may be reformulated as follows. By the Gauss *teorema egregium*, we may calculate from the given g_{ij} the Gauss curvature K. If indeed there is a minimal surface having the given metric, then, as is well known,

$$K \leqq 0 \tag{5.1}$$

at every point. If this is the case, we may form the quantities

$$\tilde{g}_{ij} = \sqrt{-K}g_{ij}, \tag{5.2}$$

and, using these, compute the corresponding Gauss curvature $\tilde{K}$ at each point where $K\neq 0$. The answer to our question may then be stated:

The given g_{ij} arise locally as the first fundamental form of a minimal surface in R^3 if and only if (5.1) *holds along with*

$$\tilde{K} \equiv 0. \tag{5.3}$$

We shall call equation (5.3) the *Ricci condition.* Pinl [38] posed the question, to what extent can the Ricci condition be expected to hold for minimal surfaces in R^n, for $n>3$? This question was completely answered by Lawson [30(c)] who obtained the following result.

THEOREM 5.1. *Let g_{ij} be the coefficients of the first fundamental form of a minimal surface M in $\boldsymbol{R}^n$. If the Ricci condition is satisfied by these g_{ij}, we know that they correspond locally to the first fundamental form of a minimal surface $\hat{M}$ in $\boldsymbol{R}^3$. Then either*

1. *M lies in $\boldsymbol{R}^3$ and belongs to a specific one-parameter family of surfaces associated to $\hat{M}$, or else*
2. *M lies in $\boldsymbol{R}^6$ and belongs to a specific two-parameter family of surfaces obtained from $\hat{M}$, none of which lie in any $\boldsymbol{R}^5$.*

COROLLARY 1. *The Ricci condition is an intrinsic condition which completely characterizes minimal surfaces lying in $\boldsymbol{R}^3$ among all minimal surfaces in $\boldsymbol{R}^4$ or $\boldsymbol{R}^5$.*

COROLLARY 2. *The set of all minimal surfaces in $\boldsymbol{R}^n$ isometric to a given minimal surface in $\boldsymbol{R}^3$ consists of a specific two-parameter family of surfaces lying in $\boldsymbol{R}^6$.*

The proof of this theorem uses results of Calabi [**9(a)**] on isometric imbeddings of complex manifolds, together with a characterization of the Ricci condition in terms of the generalized Gauss map. (See the discussion of the Gauss map in part B of this section.)

Calabi [**9(c)**] has also used his earlier results to obtain intrinsic characterizations of minimal surfaces in $\boldsymbol{R}^n$. His method is the following. Let

$$x(u): D \to \boldsymbol{R}^n \tag{5.4}$$

define a minimal surface M. For two-dimensional surfaces, we know that there always exist isothermal parameters. This means that after a reparametrization, we may assume that

$$g_{11} \equiv g_{22}, \ g_{12} \equiv 0 \qquad \text{in } D. \tag{5.5}$$

Let $\lambda^2(u)$ be the common value of g_{11} and g_{22}. Then for any function ϕ: $M \to \boldsymbol{R}$, equation (2.1) for the Laplace-Beltrami operator becomes

$$\Delta\phi = \frac{1}{\lambda^2}\left(\frac{\partial^2\phi}{\partial u_1^2} + \frac{\partial^2\phi}{\partial u_2^2}\right).$$

In other words, a function ϕ is harmonic on M if and only if it is a harmonic function in the domain D. Introduce a complex parameter

$$\zeta = u_1 + iu_2$$

in D. Then by Theorem 2.1, M is minimal if and only if each coordinate function x_k is a harmonic function $h_k(\zeta)$. Equivalently, in a neighborhood of each point of D, the map (5.4) can be represented as

$$x(u) = \operatorname{Re} \Phi(\zeta) \tag{5.6}$$

where $\Phi(\zeta) = (\Phi_1(\zeta), \cdots, \Phi_n(\zeta))$, each $\Phi_k(\zeta)$ being an analytic function.

We may note that this representation of minimal surfaces in R^n was pointed out already in 1874 by Lipschitz [**31**], who studied subvarieties of arbitrary dimension and codimension in a space with a Riemannian metric, and showed that the vanishing of the mean curvature vector was equivalent to the vanishing of the first variation of volume.

It is a simple matter to verify that the surface in R^n defined by

$$y(u) = \operatorname{Im} \Phi(\zeta) \tag{5.7}$$

is again a minimal surface, called the *adjoint* of the surface (5.6). Also for every real α, the surface defined by

$$\cos \alpha x(u) + \sin \alpha y(u)$$

is a minimal surface. This gives a one-parameter family of *associate surfaces*, which was referred to in Theorem 5.1. These surfaces are all isometric.

Consider next the surface in R^{2n} defined by

$$\frac{1}{\sqrt{2}} (x_1(u), \cdots, x_n(u), y_1(u), \cdots, y_n(u)). \tag{5.8}$$

This is again a minimal surface isometric to the original surface (5.4). But identifying R^{2n} with complex n-space C^n, we see that the surface (5.8) is the real form of the complex curve

$$\frac{1}{\sqrt{2}} \Phi(\zeta): D \to C^n. \tag{5.9}$$

We have thus shown that every minimal surface in R^n is isometric to a complex analytic curve in C^n considered as a real surface. Since, conversely, a complex analytic curve considered as a real surface is always a minimal surface, we see that characterizing all metrics which may arise from minimal surfaces in some R^n is equivalent to characterizing all those which arise from complex analytic curves in some C^N. The relation between the possible pairs of values of n and N which can arise remains to be explored, and this is done by Calabi [**9(c)**].

B. *The generalized Gauss map*. The Grassmannian $G_{2,n}$ of oriented 2-planes in $\boldsymbol{R}^n$ may be identified in various ways with the complex hyperquadric

$$Q_{n-2} = \left\{ z = (z_1, \cdots, z_n) \in P_{n-1}(\boldsymbol{C}) \;\middle|\; \sum_{k=1}^{n} z_k^2 = 0 \right\}.$$

(See, for example, Chern [**10(a)**], and Osserman [**36(a)**, (**c**)].) This identification allows us to define various structures on $G_{2,n}$ induced from the corresponding structures on complex projective space $P_{n-1}(\boldsymbol{C})$. In particular, we have a complex analytic structure, and a Riemannian metric induced by the standard Fubini-Study metric on $P_{n-1}(\boldsymbol{C})$, suitably normalized. (See, for example, [**36(c)**, §12].)

Given a two-dimensional surface M in $\boldsymbol{R}^n$, the *generalized Gauss map* is the map

$$g: M \to G_{2,n}$$

defined by $g(p) = T_p(M)$.

We shall henceforth identify $G_{2,n}$ with Q_{n-2}, and consider g as a map

$$g: M \to Q_{n-2} \subset P_{n-1}(\boldsymbol{C}). \tag{5.10}$$

In the case of a minimal surface defined by (5.6), the Gauss map turns out to be given simply by

$$g: x(\zeta) \to z(\zeta); \quad z_k = \overline{\Phi_k'(\zeta)}. \tag{5.11}$$

Using this explicit formula, one may verify directly a number of properties of the Gauss map of minimal surfaces. For example,

1. the Gauss map of a minimal surface is an antiholomorphic map;
2. the Gauss curvature at each point is the negative of the area dilation under the Gauss map;
3. the minimal surface satisfies the Ricci condition (5.3) if and only if its image under the Gauss map has constant Gauss curvature equal to 1.

Property 3 is the one referred to earlier which was used by Lawson to obtain Theorem 5.1. Properties 1 and 2 have been used to obtain various global results, such as the following.

THEOREM 5.2 (CHERN [**10(a)**], OSSERMAN [**36(a)**]). *A complete minimal surface in $\boldsymbol{R}^n$ is either a plane or else its image under the Gauss map intersects an everywhere dense set of hyperplanes in $P_{n-1}(\boldsymbol{C})$.*

THEOREM 5.3 (CHERN AND OSSERMAN [**12**]). *A complete minimal* *surface in R^n has finite total curvature if and only if its Gauss map is algebraic.*

THEOREM 5.4 (CHERN AND OSSERMAN [12]). *Let M be a complete minimal surface in R^n, not a plane, and let m be the smallest dimension of a linear subspace L of $P_{n-1}(C)$ containing $g(M)$. Then given any $m(m+1)/2$ hyperplanes of L in general position, $g(M)$ must intersect at least one of them.*

Finally, we mention a recent result of Jonker [28] which depends in part on an examination of the Gauss map of two-dimensional surfaces in R^n. Jonker studies surfaces M whose mean curvature normal is always perpendicular to a fixed $(n-2)$-dimensional linear space L. He shows that either M is a minimal surface in R^n or else that except for possible isolated points, M is locally a minimal submanifold of a cylindrical hypersurface generated by L.

C. *Nonparametric surfaces.* By Theorem 2.5, the theory of nonparametric two-dimensional surfaces coincides with the theory of solutions of the elliptic system of equations (2.5), with $m=2$. It is natural to look for analogs of the results discussed in §4 for the case of hypersurfaces. We cite several which can be found in the papers of Osserman [36(b), (c)]. In the following statements, it is understood throughout that we are dealing with the case $m=2$.

THEOREM 5.5. *Every C^2 solution of the system* (2.5) *is real analytic.*

The proof of this theorem, as well as of Theorems 5.6 and 5.7 below, uses the auxiliary equations (2.6). In the case $m=2$, they may be interpreted as exactness conditions guaranteeing the existence of a pair of new functions, which may be used as in previous proofs for $n=3$.

Theorem 4.2 goes over without change:

THEOREM 5.6. *Every entire solution of* (2.5) *which is bounded on one side is constant.*

The most obvious analog of Theorem 4.3 turns out to be false. That is, there exists a great variety of entire solutions for all $n \geqq 4$ which need not be linear nor even lie in a hyperplane. However, there is an important restriction on entire solutions. To state it, we refer to the generalized Gauss map, discussed in part B above. We say that the Gauss map is *degenerate* if the image lies in a hyperplane of $P_{n-1}(C)$.

THEOREM 5.7. *The surface defined by an entire solution of* (2.5) *always has a degenerate Gauss map.*

For $n=3$, a surface has a degenerate Gauss map if and only if it lies on a plane. Thus Theorem 5.7 contains the case $n=3$ of Theorem 4.3.

We may note that the proofs of Theorems 5.6 and 5.7 contain implicitly the following lemma, on which they depend.

LEMMA. *A one-to-one harmonic mapping defined in the whole plane must be linear.*

Concerning Theorem 4.5, it turns out that not even a weak form will hold when the codimension is greater than 1. Namely, one can construct bounded solutions of (2.5) in the case $m=2$, $n=4$, which are defined in a punctured disk, and which have a nonremovable isolated singularity.

On the other hand, Theorem 4.6 is true in its strongest form. If D is a bounded plane domain, then (2.5) has a solution for arbitrarily prescribed continuous boundary values if and only if D is convex.

D. *Other results.* Since we have a complex structure on two-dimensional surfaces, it is not surprising that Nevanlinna's theory of value distribution will play a role. In particular, for the case $n=3$, the Gauss map defines a complex analytic map of the surface into the sphere, which may be considered simply as a meromorphic function on the surface. A number of applications of Nevanlinna theory to this case are given in Osserman [**36**(a)].

Various generalizations of Nevanlinna theory also enter in. Since the original surface is not necessarily simply-connected, it is natural to try to apply the more general value-distribution theory for arbitrary Riemann surfaces. For this, see Sario and Noshiro [**41**]. For minimal surfaces in R^n, the generalized Gauss map (5.11) is the complex conjugate of an analytic curve in complex projective space. Here we have the Ahlfors-Weyl generalization of Nevanlinna theory, and it is this viewpoint which is adopted in the paper of Chern and Osserman [**12**].

Recent work of Beckenbach and Hutchison [**3**] has taken a very different approach. The Gauss map is not considered but the authors develop a new generalization of Nevanlinna theory for minimal surfaces themselves, considered as maps from the complex plane into R^3. To each point in space are associated generalizations of Nevanlinna's counting function and proximity function, as well as a new quantity called the "visibility function." This last may be interpreted as related to the way the surface is seen when viewed from the given point. The sum of these three functions describes the "total affinity" of the surface to the given point. The total affinity to the point at infinity is called the *characteristic function* of the surface, and is used to prove an analog of Nevanlinna's First Main Theorem: the total affinity of a meromorphic minimal surface to an arbitrary point differs from the characteristic function by a bounded quantity.

Other recent contributions to the theory of minimal surfaces, considered as complex analytic maps of the plane onto a surface in $\boldsymbol{R}^n$, have been made by Dinghas [**16**]. He obtains, in particular, the following analog of the Schwarz-Pick lemma. Let D be the unit disk in $\boldsymbol{R}^2$ and $x(\zeta): D \rightarrow \boldsymbol{R}^n$ a minimal surface lying in the unit ball B in $\boldsymbol{R}^n$. Suppose this surface satisfies the condition that for some constant $c<1$, the projection of the radius vector $x(u)$ into the normal space at the point $p=x(u)$ is always bounded by c. Let $[x, y]_n$ denote the distance between the points x and y with respect to the hyperbolic metric in the unit ball of $\boldsymbol{R}^n$. Then for all points ζ_1, ζ_2 in D, we have

$$[x(\zeta_1), x(\zeta_2)]_n \leqq [\zeta_1, \zeta_2]_2/\sqrt{1 - c^2}.$$

Finally, we mention the recent discovery of a number of remarkable surfaces in $\boldsymbol{R}^3$ by A. H. Schoen. These are infinite periodic minimal surfaces with no self-intersections. Among them is a surface containing no straight lines built out of an infinite number of congruent curvilinear hexagons whose sides form a family of curves which are almost, but not exactly, circular helices. This surface is associate (in the sense described in part A of this section) to a classical surface of Schwarz. A model has been constructed by Schoen, and a photograph appears in Osserman [**36**(c)]. Schoen's results, including some joint work with H. B. Lawson, have not yet been written up for publication, but a number of abstracts have appeared in the Notices of the American Mathematical Society [**42**].

6. **Minimal cones.** Let $F(x^1, \cdots, x_n)$ be a smooth function defined in some domain in $\boldsymbol{R}^n$. Let us use the notation

$$F_i = \partial F/\partial x_i$$

so that

$$\nabla F = (F_1, \cdots, F_n).$$

By the implicit function theorem, the equation

$$F(x_1, \cdots, x_n) = c \tag{6.1}$$

can be solved for one of the coordinates in terms of the other $n-1$ coordinates in some neighborhood of any point satisfying this equation at which $\nabla F \neq 0$. This means that the nonsingular points of each level set form a hypersurface in $\boldsymbol{R}^n$.

LEMMA 6.1. *At each point where $\nabla F \neq 0$, the hypersurface* (6.1) *through that point has mean curvature h given by*

$$|h| = \frac{1}{|\nabla F|^3}\left(\sum_{i,j=1}^{n} F_i F_j F_{ij} - |\nabla F|^2 \sum_{i=1}^{n} F_{ii}\right). \tag{6.2}$$

REMARKS. 1. Since the level set (6.1) has no intrinsic orientation, the sign of h depends on an arbitrary choice of unit normal. Only the absolute value of h is uniquely determined.

2. Dombrowskii [**17**] has studied in detail the problem of obtaining explicit expressions for the basic geometric entities associated with a submanifold of a Riemannian manifold, when that submanifold is defined implicitly by setting a number of functions equal to constants. Equation (6.2) is contained as a special case of one of his formulas.

PROOF. By hypothesis, some $F_k \neq 0$. Say $F_n \neq 0$. Then equation (6.1) may be solved in the form

$$x_n = f(x_1, \cdots, x_{n-1}).$$

Using the notation $f_i = \partial f/\partial x_i$, we have from

$$F(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1})) \equiv c$$

that

$$F_i + F_n f_i \equiv 0, \qquad i = 1, \cdots, n-1$$

and

$$F_{ij} + F_{in} f_j + F_{nj} f_i + F_{nn} f_i f_j + F_n f_{ij} \equiv 0, \qquad i,j = 1, \cdots, n-1.$$

From (3.6) and (3.7), we find $g = |\nabla F|^2/F_n^2$ and

$$g^{ij} = \delta_{ij} - F_i F_j / |\nabla F|^2, \qquad i,j = 1, \cdots, n-1.$$

Substituting into (3.13) yields an expression which reduces to (6.2).

COROLLARY. *The level set* (6.1) *is a minimal surface if and only if F satisfies the equation*

$$\Delta F = \sum F_i F_j F_{ij} / |\nabla F|^2. \tag{6.3}$$

A class of minimal hypersurfaces which have been most intensively studied has been the minimal cones. A *cone* in $\boldsymbol{R}^n$ with vertex at the origin is a union of rays through the origin, and is uniquely determined by its intersection with the unit sphere S^{n-1}. It is easy to verify that a cone will be a minimal hypersurface in $\boldsymbol{R}^n$ if and only if its intersection with S^{n-1} is a minimal hypersurface of S^{n-1}. (This is clear using the fact that for a manifold lying on S^{n-1}, its mean curvature vector relative to S^{n-1} is simply the projection on the tangent space to S^{n-1} of its mean curvature vector relative to $\boldsymbol{R}^n$.) Thus the theory of minimal cones in $\boldsymbol{R}^n$ is equivalent to the theory of minimal submanifolds of S^{n-1}. We shall use this equivalence freely in the discussion below.

A large part of the impetus for the study of minimal cones comes from the measure-theoretic approach to minimal varieties, where they play a major role. One of the chief problems is to find conditions under which a minimal cone has no singularity at the origin; i.e., the cone is in fact a linear space. At the other extreme, one may ask just how complicated a minimal cone can be. We start by looking at the situation in low dimensions.

First, we note that the reason minimal cones were not considered classically is that in $\boldsymbol{R}^3$ the only minimal cones are planes through the origin. Namely, a cone in $\boldsymbol{R}^3$ has Gauss curvature identically zero, and if its mean curvature is also zero, then it must be a plane.

In $\boldsymbol{R}^4$, there are nontrivial minimal cones. For example, the equation

$$x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0 \tag{6.4}$$

defines a minimal cone, as one verifies immediately from equation (6.3). Its intersection with S^3 has the topological type of a torus, and in fact is isometric to a square in $\boldsymbol{R}^2$ with opposite sides identified, as one sees from the explicit mapping

$$x_1 = \frac{1}{\sqrt{2}} \cos u_1, \quad x_2 = \frac{1}{\sqrt{2}} \sin u_1, \quad x_3 = \frac{1}{\sqrt{2}} \cos u_2, \quad x_4 = \frac{1}{\sqrt{2}} \sin u_2,$$
$$0 \leqq u_1 \leqq 2\pi, \quad 0 \leqq u_2 \leqq 2\pi.$$

This is the so-called *Clifford torus*. For a long time, it defined the only known nontrivial minimal cone in $\boldsymbol{R}^4$. Then the following results were obtained.

THEOREM 6.1 (ALMGREN [2(a)]). *If a minimal cone in $\boldsymbol{R}^4$ intersects S^3 in a surface which is topologically a 2-sphere, then the cone must be a hyperplane.*

Thus, a nontrivial minimal cone in $\boldsymbol{R}^4$ cannot be "near" a hyperplane, but must be reasonably complicated.

THEOREM 6.2 (LAWSON [**30**(a), (e), (f)]). *For every positive integer g, there exists a minimal cone in R^4 whose intersection with S^3 is a compact surface of genus g.*

Thus there exist minimal cones in R^4 of arbitrarily complicated type. Lawson's proof depends on a geometric construction, and it has not yet been possible to find explicit equations for the surfaces of higher genus.

In higher dimensions, a large number of new examples of minimal cones was found by Hsiang [**26**(b)], using Lie group methods. He also considers the problem of finding *algebraic* minimal cones, obtained by setting a homogeneous polynomial equal to zero. For quadratic polynomials, there were the known examples,

$$q(x_1^2 + \cdots + x_{p+1}^2) - p(x_{p+2}^2 + \cdots + x_{p+q+2}^2) = 0, \qquad p \geqq 1,\ q \geqq 1,\ p + q + 2 = n, \tag{6.5}$$

generalizing (6.4). That these define minimal cones is again a direct consequence of equation (6.3). The intersection of (6.5) with the sphere S^{n-1} is a compact surface homeomorphic to $S^p \times S^q$. Hsiang showed that these are in fact the only algebraic minimal cones of degree 2.

The cones (6.5) also have an intrinsic geometric characterization among all minimal cones, which can most easily be stated in terms of their intersection with the sphere S^{n-1}. This intersection is a manifold having at each point a *scalar curvature* defined as an average of all sectional curvatures at the point. The following result was proved independently by Lawson [**30**(b)] and by Chern, do Carmo, and Kobayashi [**11**].

THEOREM 6.3. *A minimal hypersurface of S^{n-1} having constant scalar curvature equal to $(n-4)/(n-3)$ must be (up to rotations of S^{n-1}) an open subset of the intersection of S^{n-1} with one of the cones* (6.5).

This result takes on particular interest in view of the following theorem of Simons [**44**].

THEOREM 6.4. *If a compact minimal hypersurface of S^{n-1} has scalar curvature κ satisfying everywhere $\kappa \geqq (n-4)/(n-3)$, then κ is constant, and either $\kappa \equiv (n-4)/(n-3)$ or $\kappa \equiv 1$.*

For a minimal hypersurface of a sphere, the scalar curvature always satisfies $\kappa \leqq 1$, and $\kappa \equiv 1$ holds only for an equatorial hypersphere; i.e., the intersection of the sphere with a hyperplane through the origin. Thus, combining Theorems 6.3 and 6.4 yields

COROLLARY. *If the scalar curvature κ of a compact minimal hypersurface M of S^{n-1} lies in the interval*

$$(n-4)/(n-3) \leqq \kappa \leqq 1,$$

then M is the intersection of S^{n-1} either with a hyperplane or (up to a rotation) with one of the cones (6.5).

In particular, a compact minimal hypersurface of S^{n-1} cannot have constant scalar curvature lying in the open interval $(n-4)/(n-3) < \kappa < 1$. It would be interesting to know what values of κ can actually occur as the constant scalar curvature of compact minimal hypersurfaces. In the case $n=4$, a hypersurface of S^3 is a two-dimensional surface, the scalar curvature reduces to the Gauss curvature, and we have the following strong result.

THEOREM 6.5 (LAWSON [**30(b)**]). *If a minimal surface M in S^3 has constant Gauss curvature K, then either $K=1$ and M lies on an equatorial sphere or else $K=0$ and M lies on the Clifford torus.*

For further results in this direction, see Otsuki [**37**].

Returning to Simon's result, Theorem 6.4, we note that, like Theorem 6.1, it may be interpreted as saying that if a compact minimal hypersurface is in a certain sense close to being an equatorial sphere, then it must in fact be one. In Theorem 6.1, for the sphere S^3, it was sufficient to assume that the hypersurface was topologically a sphere. It is not known whether the same result holds in higher dimensions. Simons' condition is that the scalar curvature of the hypersurface be sufficiently close to that of an equatorial sphere.

Still a third theorem of this type is due to de Giorgi [**13**].

THEOREM 6.6. *A compact minimal hypersurface of a sphere whose normals all lie in an open hemisphere must be an equatorial hypersphere.*

REMARKS. 1. If we consider the cone over the given hypersurface of the sphere, it will have the same set of normals. The condition that the normals lie in an open hemisphere is equivalent to assuming that the cone may be represented in nonparametric form under a suitable rotation of coordinates. The conclusion is that the cone is a hyperplane. This is the form in which de Giorgi stated and applied the theorem.

2. The above form of the theorem is due to Simons [**44**] who gave a different proof than de Giorgi, and generalized it to compact minimal surfaces of higher codimension in the sphere. Still another proof was given by Reilly [**39**], who further sharpened Simons' results. He

showed in particular that if a compact minimal submanifold of S^{n-1} has the property that its normal space at each point makes an angle of less than $\cos^{-1}\sqrt{\frac{2}{3}}$ with a fixed space of the same dimension, then it must be an equatorial submanifold; i.e., the intersection of S^{n-1} with a linear subspace of R^n.

3. The above results may be viewed as statements about the Gauss map of compact minimal submanifolds of a sphere. They appear as the natural analog of Theorem 5.2 on the Gauss map of complete minimal surfaces in Euclidean space. In both cases, the conclusion is that if the Gauss map is sufficiently restricted, then it must be constant.

We turn finally to another important characterization of hyperplanes among minimal cones.

THEOREM 6.7 (SIMONS [44]). *Let M be a minimal cone in R^n whose intersection with S^{n-1} is a compact hypersurface of S^{n-1}. If $n \leqq 7$, then either M is a hyperplane or else there exists a variation of the part of M inside S^{n-1}, keeping the boundary fixed, which decreases the $(n-1)$-dimensional volume.*

This result is one of the major breakthroughs in Simons' paper. Combined with earlier work of Fleming, de Giorgi, and Almgren, it settled two major questions. One is the regularity of minimal hypersurfaces in R^n, for $n \leqq 7$, which solve Plateau's problem. The other is Bernstein's Theorem (Theorem 4.3 above).

Simons' proof depends on a close analysis of the formula for the second variation, and on an important formula which he derives for the Laplacian of the second fundamental form of a minimal variety. Simons further showed that the value $n=7$ was the best possible. Namely, the minimal cone (6.5) with $p=q=3$ in R^8 is in fact stable relative to its intersection with S^7; i.e., every variation of the part inside S^7, keeping the boundary fixed, initially increases volume.

The question remained whether this example of Simons actually provided an absolute minimum of volume for the given boundary. This question was answered in the affirmative by Bombieri, de Giorgi, and Giusti [7]. This provided an example in R^8 where the solution to the Plateau problem is *not* everywhere regular, and it further provided the impetus for the authors' result, Theorem 4.4 above, settling the Bernstein problem in all dimensions.

Thus the study of minimal cones has led to the solution of two of the major open problems concerning minimal varieties in Euclidean space.

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