ANALYTIC SHEAVES OF LOCAL COHOMOLOGY

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Communicated by Armand Borel, April 18, 1969

Suppose $\mathfrak{F}$ is a coherent analytic sheaf on a complex analytic space X. Denote by $S_k(\mathfrak{F})$ the analytic subvariety $\{x \in X \mid \operatorname{codh} \mathfrak{F}_x \leqq k\}$. For any open subset D of X, denote by $\bar{S}_k(\mathfrak{F} \mid D)$ the topological closure of $S_k(\mathfrak{F} \mid D)$ in X. If V is an analytic subvariety of X, denote by $\mathfrak{H}_V^k(\mathfrak{F})$ the sheaf defined by the presheaf $U \mapsto H_V^k(U, \mathfrak{F})$, where $H_V^k(U, \mathfrak{F})$ is the k-dimensional cohomology group of U with coefficients in $\mathfrak{F}$ and supports in V. If $\phi: X \to Y$ is a holomorphic map, denote by $\phi_k(\mathfrak{F})$ the kth direct image of $\mathfrak{F}$ under ϕ. If X, $\mathfrak{F}$, and V are complex algebraic instead of analytic, $\mathfrak{H}_V^k(\mathfrak{F})$ has the same meaning and $\mathfrak{F}^h$ denotes the coherent analytic sheaf canonically associated with $\mathfrak{F}$.

Our results are as follows:

THEOREM A. *Suppose V is an analytic subvariety of a complex analytic space $(X, \mathfrak{H})$, q is a nonnegative integer, and $\mathfrak{F}$ is a coherent analytic sheaf on X. Let $\theta: X - V \to X$ be the inclusion map. Then the following three statements are equivalent:*

(i) *$\theta_0(\mathfrak{F} \mid X - V), \cdots, \theta_q(\mathfrak{F} \mid X - V)$ (or equivalently $\mathfrak{H}_V^0(\mathfrak{F}), \cdots, \mathfrak{H}_V^{q+1}(\mathfrak{F})$) are coherent on X.*

(ii) *For every $x \in V$, $\theta_0(\mathfrak{F} \mid X - V)_x, \cdots, \theta_q(\mathfrak{F} \mid X - V)_x$ (or equivalently $\mathfrak{H}_V^0(\mathfrak{F})_x, \cdots, \mathfrak{H}_V^{q+1}(\mathfrak{F})_x$) are finitely generated over $\mathfrak{H}_x$.*

(iii) *$\dim V \cap \bar{S}_{k+q+1}(\mathfrak{F} \mid X - V) < k$ for every $k \geqq 0$.*

THEOREM B. *Suppose V is an algebraic subvariety of a complex algebraic space X, q is a nonnegative integer, and $\mathfrak{F}$ is a coherent algebraic sheaf on X. Then $\mathfrak{H}_V^0(\mathfrak{F}), \cdots, \mathfrak{H}_V^{q+1}(\mathfrak{F})$ are coherent algebraic sheaves on X if and only if $\mathfrak{H}_V^0(\mathfrak{F}^h), \cdots, \mathfrak{H}_V^{q+1}(\mathfrak{F}^h)$ are coherent analytic sheaves on X. If so, the canonical homomorphisms $\mathfrak{H}_V^k(\mathfrak{F})^h \to \mathfrak{H}_V^k(\mathfrak{F}^h)$ are isomorphisms for $0 \leqq k \leqq q+1$.*

In the theory of extending coherent analytic sheaves, the main problem is to answer the following question: Suppose $\mathfrak{F}$ is a coherent analytic sheaf on $X - V$, where V is an analytic subvariety of a complex analytic space X. Let $\theta: X - V \to X$ be the inclusion map. When is $\theta_0(\mathfrak{F})$ coherent? This question has been answered in various ways in [**1**] through [**10**]. Theorem A gives a criterion for the coherence of $\theta_q(\mathfrak{F})$ after a coherent analytic extension has been found. This criterion given in Theorem A sharpens a result of Trautmann [**11**]. Theorem B answers in the affirmative a question raised by Serre [**2**, pp. 373–374].

The proofs consist of refining the techniques in [11] and skillfully making use of results concerning gap-sheaves and homological codimensions and zero-divisors of stalks of coherent sheaves. There is an algebraic analog of the same formulation for Theorem A. Details will appear elsewhere.

ADDED IN PROOF. In a paper to be published, Trautmann independently has also obtained the equivalence of (i) and (iii) of Theorem A.

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