A THEOREM OF STOUT

BY MYRON GOLDSTEIN[1] AND T. R. CHOW[2]

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In a paper that appeared in Mathematische Annalen in 1968, E. L. Stout proved a generalization of a theorem of Rado which states that if f is continuous on the set $D=\{z: |z|<1\}$ and holomorphic on $D-f^{-1}(0)$, then f is holomorphic on D. Stout's generalization is as follows:

THEOREM. *Let E be a set of capacity zero in the complex plane, and let E_0 be a relatively closed set in D. If f is a function bounded and holomorphic in $D-E_0$, if f' does not vanish identically, and if for every sequence $\{z_n\}$ in $D-E_0$ such that $z_n \to z_0 \in E_0$ and $\lim f(z_n)=w_0$ exists it is the case that $w_0 \in E$, then f is holomorphic throughout D.*

We shall generalize this theorem by replacing D by an arbitrary hyperbolic Riemann surface and replacing the assumption that f is bounded by the assumption that f belong to the Hardy class H^P, $0<P\leqq \infty$, i.e., that $|f|^P$ possesses a harmonic majorant. Thus our theorem reads as follows:

THEOREM. *Let E be a set of capacity zero in the complex plane, and let E_0 be a closed subset of a hyperbolic Riemann surface R. If $f \in H^P(R-E_0)$, i.e., f is of class H^P on each component of $R-E_0$, $0<P\leqq\infty$, if f is nonconstant on some component of $R-E_0$, and if for every sequence $\{a_n\}$ in $R-E_0$ such that $a_n \to a_0 \in E_0$ and $\lim f(a_n)=b_0$ exists it is the case that $b_0 \in E$, then f is holomorphic throughout D.*

REMARK. In the proof we shall use the same notation and terminology in [1] and [2].

PROOF. Let $S=R-E_0$ and let S_0 be a component of S on which f is nonconstant. It then follows that S_0 is hyperbolic and that its universal covering surface is D. Let π denote the projection map of D onto S_0. Since π is a Fatou mapping, the fine limit function $\hat{\pi}$ is defined a.e. on ∂D. Here a.e. refers to Lebesgue measure. Since D is a regular covering surface of S_0, it follows that every point P' in S_0 has a neighborhood V with the property that each component of $\pi^{-1}(V)$ is compact. Hence π is of Blaschke type. It follows that for a.e. point

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$b \in \partial D$, $\hat{\pi}(b)$ is defined and lies in the minimal Martin boundary of S_0. Let us now regard π as a mapping of D into R. π is still a Fatou mapping and hence $\hat{\pi}$ is defined a.e. on ∂D. Consequently for a.e. $b \in \partial D$, $\hat{\pi}(b)$ is defined and either lies in Δ_1, the minimal Martin boundary of R, or else lies in E_0. Since $f \in H^p(S_0)$, f is a Fatou mapping and hence $f \circ \pi$ is a Fatou mapping. Hence $(f \circ \pi)^\wedge$ is defined a.e. on ∂D. If the set of points $b \in \partial D$ where $\hat{\pi}(b)$ is defined and lies in E_0, is of positive Lebesgue measure, then the set of points $f(\hat{\pi}(b)) = (f \circ \pi)^\wedge(b)$ would have to be of positive capacity, since f is nonconstant, by a theorem of Doob; but on the other hand it would have to be of capacity zero since by hypothesis this set would have to be contained in E, and E has capacity zero. Hence π regarded as a mapping of D into R is of Blaschke type and hence by a theorem of Heins, E_0 is a set of capacity zero. Hence by a theorem of Parreau, f has a holomorphic extension to R.

REMARK. Actually, more is true. Thus $R - E_0$ is connected and the extended function is of class H^p on R.

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OREGON STATE UNIVERSITY, CORVALLIS, OREGON 97331