THE SPECTRUM OF NONCOMPACT G/Γ AND THE COHOMOLOGY OF ARITHMETIC GROUPS

BY HOWARD GARLAND[1]

Communicated by Louis Auslander, February 13, 1969

Introduction. The purpose of this note is to announce a theorem in the representation theory of semisimple groups (Theorem 1.2, below). This theorem implies that certain spaces of square summable harmonic forms on noncompact locally symmetric spaces, associated with $\boldsymbol{Q}$-rank one arithmetic groups, are finite dimensional. Assertion (1.3) then gives information about the boundary behavior at ∞ of such forms. Combining (1.3) with the computations in [4] and Raghunathan's square summability criterion in [6], we obtain upper bounds for some betti numbers of locally symmetric spaces associated with $\boldsymbol{Q}$-rank one arithmetic groups (these spaces are noncompact, but have the homotopy type of a finite simplicial complex (see [7])). In some cases we obtain vanishing theorems for the first and second betti numbers. For the first betti number, such a vanishing theorem was obtained in greater generality by D. A. Kazdan (see [3]) by a different method. We remark that Raghunathan's square summability criterion has been generalized to arbitrary $\boldsymbol{Q}$-rank in [1]. Therefore an extension of Theorem 1.2 to arbitrary $\boldsymbol{Q}$-rank would yield a corresponding extension of our present results on cohomology. A detailed proof of Theorem 1.2 and a full discussion of the application of this theorem to the cohomology of arithmetic groups will appear elsewhere. I wish to express my thanks to S. T. Kuroda and M. S. Raghunathan for stimulating discussions.

We now introduce some notation. Let $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ denote the fields of rational, real, and complex numbers, respectively, and let $\boldsymbol{Z}$ denote the ring of rational integers. Let $\mathbf{G}$ denote a connected, linear, semisimple, algebraic group which is defined and simple over $\boldsymbol{Q}$. For a subring $A \subset \boldsymbol{C}$, let $\mathbf{G}_A$ denote the A-rational points of $\mathbf{G}$. However, when $A = \boldsymbol{R}$, we let $G = \mathbf{G}_{\boldsymbol{R}}$. We let $\mathfrak{g}$ denote the Lie algebra of G, $\mathfrak{g}_C$ the complexification of $\mathfrak{g}$, and $\mathfrak{G}$ the universal enveloping algebra of $\mathfrak{g}_C$. We make the convention that $\mathfrak{g}$ is the space of right invariant vector fields on G. Hence $\mathfrak{G}$ is the space of right invariant differential operators on G. We denote the center of $\mathfrak{G}$ by $\mathfrak{Z}$. As is well known, $\mathfrak{Z}$ may be identified with the space of (adjoint-)invariant polynomials

[1] The author was partially supported by NSF Grant GP-7131 and a Yale University Junior Faculty Fellowship.

on $\mathfrak{g}_C$. In particular, there is a unique element $\Delta_G \in \mathfrak{Z}$, called the Casimir operator, which corresponds to the Killing form under this identification.

Let $\Gamma \subset G_Q$ be an arithmetic subgroup. We fix a Haar measure dv on G, and note that dv induces a G-invariant measure on G/Γ (which we again denote by dv). We let $L_2 = L_2(G/\Gamma)$ denote the space of C^∞, $\boldsymbol{C}$-valued functions f on G/Γ, such that

$$\int_{G/\Gamma} f(x) f^{-}(x) dv(x) < \infty$$

(where "$-$" denotes complex conjugation).

We fix a maximal $\boldsymbol{Q}$-split torus ${}_Q\boldsymbol{S} \subset \boldsymbol{G}$, and we let ${}_QA$ denote the topological identity component of the $\boldsymbol{R}$-rational points of ${}_Q\boldsymbol{S}$. We let $Z({}_Q\boldsymbol{S})$ denote the centralizer of ${}_Q\boldsymbol{S}$ in $\boldsymbol{G}$, and we let X_Q denote the $\boldsymbol{Q}$-rational characters of $Z({}_Q\boldsymbol{S})$. We then define $\boldsymbol{M} \subset Z({}_Q\boldsymbol{S})$ by

$$\boldsymbol{M} = \bigcap_{\chi \in X_Q} \text{kernel } \chi^2.$$

$Z({}_Q\boldsymbol{S})$ is known to have an almost direct product decomposition $Z({}_Q\boldsymbol{S}) = \boldsymbol{M}_Q\boldsymbol{S}$, and $Z({}_QA)$, the centralizer of ${}_QA$ in G, a *direct product* decomposition

$$Z({}_QA) = M_QA,$$

where M denotes the $\boldsymbol{R}$-rational points of $\boldsymbol{M}$.

We now fix a maximal compact subgroup $K \subset G$, such that K and ${}_QA$ have Lie algebras which are orthogonal with respect to the Cartan-Killing form of $\mathfrak{g}$. Let V be a finite dimensional, complex vector space with a positive definite, Hermitian inner product. Then let $\sigma: K \to \text{Aut } V$ be a representation of K which is unitary with respect to the given inner product. We let d_σ denote the complex dimension of V and we let ξ_σ denote the character of σ.

We then define a subspace L_2^σ of L_2, by

$$(0.1) \qquad L_2^\sigma = \left\{ f \in L_2 \,\middle|\, d_\sigma \int_K \xi_\sigma(k) f(k^{-1}x) dk = f(x),\ x \in G/\Gamma \right\},$$

where dk denotes Haar measure on K, normalized so that

$$\int_K dk = 1.$$

We remark that functions on G/Γ may be identified with Γ-invariant functions on G. We will make this identification whenever convenient and we will denote corresponding functions on G and G/Γ by the same letter.

1. **Statement of the main theorem.** For $\nu \in \boldsymbol{C}$, let

$$\mathfrak{G}_\nu^\sigma = \{f \in L_2^\sigma \mid \Delta_G f = \nu f\}.$$

LEMMA 1.1. *Assume $\boldsymbol{G}$ has $\boldsymbol{Q}$-rank one; i.e.* dim ${}_{\boldsymbol{Q}}\boldsymbol{S} = 1$. *Then there exists a real number J so that if $\mathfrak{G}_\nu^\sigma \neq \{0\}$, then ν is real and $\nu < J$.*

THEOREM 1.2 (MAIN THEOREM). *Assume $\boldsymbol{G}$ has $\boldsymbol{Q}$-rank one. For $c \in \boldsymbol{R}$, let*

$$\mathfrak{F}_c^\sigma = \oplus_{\nu > c} \mathfrak{G}_\nu^\sigma.$$

Then $\mathfrak{F}_c^\sigma$ is finite dimensional. Moreover, if $\nu \in \boldsymbol{R}$, $f \in \mathfrak{G}_\nu^\sigma$ and $\Lambda \in \mathfrak{S}$, we have $\Lambda f \in L_2$. If ν_1, $\nu_2 \in \boldsymbol{R}$, $f_l \in \mathfrak{G}_{\nu_l}^\sigma$ $(l = 1, 2)$, and Λ_1, $\Lambda_2 \in \mathfrak{S}$, then for $X \in \mathfrak{g}$, we have

$$\int_{G/\Gamma} (X\Lambda_1 f_1)(\Lambda_2 \bar{f}_2)dv = -\int_{G/\Gamma} (\Lambda_1 f_1)(X\Lambda_2 \bar{f}_2)dv. \tag{1.3}$$

The following is an immediate consequence of Lemma 1.1 and Theorem 1.2.

COROLLARY 1.4. *The eigenvalues of Δ_G in L_2^σ have no finite point of accumulation.*[2]

2. **An indication of the proof of the main theorem.** In this section we assume $\boldsymbol{G}$ has $\boldsymbol{Q}$-rank one. Let $\boldsymbol{P} \subset \boldsymbol{G}$ be a minimal $\boldsymbol{Q}$-parabolic subgroup and let P denote the $\boldsymbol{R}$-rational points of $\boldsymbol{P}$. We let $\boldsymbol{U}$ denote the unipotent radical of $\boldsymbol{P}$ and U the $\boldsymbol{R}$-rational points of $\boldsymbol{U}$. After conjugating $\boldsymbol{P}$ by a suitable point in $G_{\boldsymbol{Q}}$, we can assume

$$\boldsymbol{P} = \boldsymbol{M}_{\boldsymbol{Q}}\boldsymbol{S}\boldsymbol{U}, \qquad P = M_{\boldsymbol{Q}}AU.$$

We let Ξ denote a set of double coset representatives for $P_{\boldsymbol{Q}} \backslash G_{\boldsymbol{Q}}/\Gamma$, and we let

$$\Gamma_\infty = \bigcap_{q \in \Xi} q\Gamma q^{-1} \cap U.$$

U/Γ_∞ is compact, and we can therefore fix a Haar measure du on U so that $\int_{U/\Gamma_\infty} du = 1$. For $f \in L_2$ and $q \in \Xi$, we define f_q by $f_q(x) = f(xq)$, $x \in G$ (f here being identified with a right Γ invariant func-

---

[2] At first we proved $\mathfrak{G}_\nu^\sigma$ finite dimensional. We thank R. P. Langlands for pointing out that our argument also gives the finite dimensionality of $\mathfrak{F}_c^\sigma$, and hence Corollary 1.4.

tion on G). We then define f'_q by

$$f'_q(x) = \int_{U/\Gamma_\infty} f_q(xu)du, \qquad x \in G.$$

From now on, we assume $f \in \mathcal{G}^\sigma_\gamma$ for some $\nu \in R$ and some σ. In particular, $f \in L_2^\sigma$ and this means that f is a component of a V-valued, left K equivariant function. The same is then true of f'_q. Moreover, since G has the generalized Iwasawa decomposition

$$G = KM_Q AU,$$

and since f'_q is also right U invariant, we see that f'_q is uniquely determined by its restriction to $M_Q A$. We denote this restriction again by f'_q.

Recall that $M_Q A$ is a direct product. We can therefore regard f'_q as a function of two variables (the M-variable and the ${}_QA$-variable). A central step in proving Lemma 1.1 and Theorem 1.2, is to determine the nature of f'_q as a function of the ${}_QA$-variable. For we can then apply the theory of cusp forms (see [2, Chapter 1]) together with arguments from the theory of elliptic operators (see [5]) to obtain the desired results. We will describe f'_q as a function in the ${}_QA$-variable presently, but in preparation, we introduce some notation.

We let $\pi: MU \to M$ denote the natural projection. We let

$$\Gamma_P = \bigcap_{q \in \Xi} (q\Gamma q^{-1} \cap MU), \quad \text{and} \quad \Gamma_M = \pi(\Gamma_P).$$

For each $a \in {}_QA$, we set $f'_{q,a}(m) = f'_q(ma)$, $m \in M$. $f'_{q,a}$ is then a right Γ_M-invariant function on M. Moreover, Γ_M is a discrete subgroup of M and M/Γ_M is compact. Hence $f'_{q,a}$ may be regarded as a function on the compact quotient space M/Γ_M. We let $K_M = \pi(K \cap MU)$ and we define $\sigma_M: K_M \to \mathrm{Aut}\ V$, by

$$\sigma_M(\pi(k)) = \sigma(k), \qquad k \in K \cap MU.$$

We then fix a Haar measure dm on M, and define $L_2(M/\Gamma_M)$ and $L_2^{\sigma_M}(M/\Gamma_M)$ just as we did $L_2(G/\Gamma)$ and $L_2^\sigma(G/\Gamma)$, respectively. We note that $f'_{q,a} \in L_2^{\sigma_M}(M/\Gamma_M)$, for all $a \in {}_QA$. The pair $({}_QA, U)$ determines an order on the roots of ${}_QA$. We then let α denote the unique simple root and ${}_Qg$ one half the sum of the positive roots. The behaviour of f'_q as a function in a, $a \in {}_QA$, is then given by

LEMMA 2.1. *There is an orthonormal basis* $\phi_1, \cdots, \phi_l, \cdots$ *of* $L_2^{\sigma_M}(M/\Gamma_M)$, *a sequence of real numbers* $m_1, \cdots, m_l, \cdots$ *such that*

$\text{Limit}_{l\to\infty} m_l = \infty$, and a positive number λ depending only on $\mathfrak{g}$, so that if $\nu \in \mathbf{C}$ and $\mathcal{G}_\nu^\sigma \neq \{0\}$, then $\nu \in \mathbf{R}$ and there is a finite subsequence $\phi_{i_1}, \cdots, \phi_{i_N}$, with $m_{i_j} + \nu > 0, j = 1, \cdots, N$, so that if $\kappa_j = \lambda^{-1}(m_{i_j} + \nu)^{1/2}$ (here we take the positive square root), then for all $f \in \mathcal{G}_\nu^\sigma$, $q \in \Xi$, we can find $b_1, \cdots, b_n \in \mathbf{C}$, so that

$$\exp({}_Q g(\log a)) f_q'(ma) = \sum_{j=1}^{N} b_j \exp(\kappa_j \alpha(\log a)) \phi_{i_j}(m), \quad a \in {}_Q A,\ m \in M.$$

Here log *a is the unique element in the Lie algebra of ${}_Q A$ which exponentiates to a.*

REMARK. The ϕ_i and m_i are respectively the eigenfunctions and corresponding eigenvalues of a certain (essentially) elliptic invariant differential operator on $L_2^{\sigma M}(M/\Gamma_M)$ associated with Δ_G.

BIBLIOGRAPHY

1. H. Garland and W. C. Hsiang, *A square integrability criterion for the cohomology of arithmetic groups*, Proc. Nat. Acad. Sci. U.S.A. **59** (1968), 354–360.

2. Harish-Chandra, *Automorphic forms on semisimple Lie groups*, Lecture Notes in Math., vol. 62, Springer-Verlag, New York, 1968.

3. D. A. Kazdan, *On the connection of the dual space of a group with the structure of its closed subgroups*, Funkcional. Anal. i Priložen. **1** (1967), 71–74 = Functional Anal. Appl. **1** (1967), 63–65.

4. Y. Matsushima, *On Betti numbers of compact, locally symmetric Riemannian manifolds*, Osaka J. Math. **14** (1962), 1–20.

5. R. Narasimhan, *Lectures on topics in analysis*, Mimeographed Notes, Tata Institute of Fundamental Research, Bombay, 1965.

6. M. S. Raghunathan, *Cohomology of arithmetic subgroups of algebraic groups*. II, Ann. of Math. (2)**87**(1968), 279–304.

7. ———, *A note on quotients of real algebraic groups by arithmetic subgroups*, Invent. Math. **4** (1968), 318–335.

YALE UNIVERSITY, NEW HAVEN, CONNECTICUT 06520