surely to the probability that the random walk will never return to the origin. The chapter on discrete parameter Markov chains is remarkable for its elegant and compact presentation of limit theorems, including Orey's theorem, the most important recent result in the theory now seemingly essentially completed. This theorem reads: If $p^{(n)}(i, j)$ is the n-step transition probability matrix of an irreducible (=indecomposable) recurrent aperiodic Markov chain, then for any i, k

$$\lim_{n\to\infty} \sum_j \left| p^{(n)}(i,j) - p^{(n)}(k,j) \right| = 0.$$

The main limit theorem of Kolmogorov: $p^{(n)}(i, j)$ converges as $n\to\infty$, easily follows. There is a chapter on Brownian motion containing a recent result of Dworetski, Erdös and Kakutani that almost every Brownian path is nowhere differentiable. There is a substantial chapter on invariance theorems, including Donsker's invariance principle, the Doob-Donsker proof of the Kolmogorov-Smirnov theorem and finally a general invariance principle of Skorokhod, the proof of which is only sketched. The author has nearly all the material necessary to establish the recent important result of V. Strassen on the law of iterated logarithm, but he stops short of it. The last chapters are concerned with continuous parameter stochastic processes: martingales, Poisson processes, Markov processes, diffusion. They seem to present a good introduction into a field in which a great deal of research is now being done, by some of the leading probabilists. A parenthetical remark: Breiman's book, unlike Chung's, has an excellent table of contents from which the reader may for himself find out what the book contains.

In conclusion, both works under review can be highly recommended as first-year graduate textbooks; even though in this role we would give a slight edge to Chung's book, a very carefully and lucidly written account by a master of the subject, and a product, the author tells us, of many years of teaching. Breiman's book may be preferred by an instructor who wishes to rapidly introduce a very good class into modern ideas more general than independence; furthermore, a specialist will want to read this book for the rich account it gives of recent developments in fields other than his own. Each book constitutes a very valuable addition to the literature.

LOUIS SUCHESTON

An introduction to harmonic analysis, by Yitzhak Katznelson. Wiley, New York, 1968. 266 pp. $12.95.

This is a text, on the modern theory of Fourier series and integrals, designed to be used by students who know the basics of real and complex variables. (Since some of the proofs are terse, their knowledge will be well tested.) Rather than merely saying that the book is good, which it is, we prefer to discuss what it contains.

Let us agree on some notations. The letters T, Z, and R denote the circle-group, the integers, and the real-line respectively. Following the usual conventions of ambiguity, T is sometimes identified with the interval $[0, 2\pi)\subset R$. To apply complex-variable methods, we identify T with the circle $\{|z|=1\}$; then if $f\in L^1(T)$, $f(re^{i\theta})$ denotes the unique harmonic function on the unit disk whose "boundary values" (in the sense of mean convergence) coincide with $f(e^{i\theta})$. In discussing spectral synthesis, $A(T)$, $A(Z)$, and $A(R)$ are the rings of Fourier transforms of functions in $L^1(Z)$, $L^1(T)$, and $L^1(R)$ respectively. "LCA" means "locally compact abelian." The translate $f(t-a)$ of $f(t)$ is denoted by f_a.

The present text differs from Rudin's *Fourier analysis on groups*, in proving most theorems only for the special case $G=T$ or $G=R$. On the other hand, much important material not pertinent to general LCA groups finds its way into this book. Since we cannot mention everything, I have chosen to stress some topics which are fairly new, or have recently acquired a new interest.

Chapter 1 covers the basic preliminaries. Chapter 2 is concerned with pointwise convergence. Some of the results here are due to Kahane and the author: specifically the following theorem.

Let B, $C(T)\subseteq B\subseteq L^1(T)$, be a Banach space of functions on T such that:

(i) if $f\in B$, then $f_a\in B$ and $\|f_a\|=\|f\|$ (where $f_a(t)\equiv f(t-a)$);

(ii) $\|f_a-f\|\to 0$ as $a\to 0$ in T;

(iii) if $f\in B$, then $e^{int}f\in B$ and $\|e^{int}f\|=\|f\|$ $(\forall n\in Z)$.

A set $E\subset T$ is called a *set of divergence* for B, if there is some $f\in B$ whose Fourier series diverges at all points of E. Then

(a) every set of Lebesgue measure zero is a set of divergence for B;

(b) any countable union of sets of divergence is a set of divergence.

From this it follows easily that either

(1) T is a set of divergence, or

(2) all sets of divergence are of measure zero.

For $B=L^1(T)$, the alternative (1) holds (Kolmogorov (1926)). For $B=L^p(T)$, $p>1$, (2) holds (Carleson (1966) for $p=2$, Hunt (1967) for all L^p).

We note that (a) above provides a converse to the Carleson-Hunt theorem. The author omits the proof of this last-mentioned theorem, observing that it is "still rather complicated."

Chapter 3 deals with the conjugate function $\tilde{f}(e^{i\theta})$, $e^{i\theta}\in T$. This name, of course, is derived from the fact that $\tilde{f}(re^{i\theta})$ is the harmonic conjugate of $f(re^{i\theta})$. Carleson's proof of the Lusin conjecture, referred to above, is based on the inequalities of Hardy-Littlewood and M. Riesz which are developed in this chapter. The Hardy-Littlewood maximal theorem gives an estimate for the L^p-norm of $f^*(e^{i\theta}) = \sup_{0<r<1}|f(re^{i\theta})|$ (where r may vary with θ) in terms of $\|f(e^{i\theta})\|_p$ ($p\neq 1, \infty$). The Riesz theorem states that the mapping $f\rightarrow\tilde{f}$ is bounded in L^p, $1<p<\infty$. In both cases, modifications are necessary for L^1 and L^∞. All of the results in Chapter 3 are "classical," and can be found e.g. in Zygmund's well known treatise, but many of the proofs have been simplified.

Chapter 4 presents the Riesz-Thorin convexity theorem, and as a corollary, the Hausdorff-Young theorem, following the standard complex-variables approach. Chapter 5 is on lacunary Fourier series, with references to Sidon sets. Chapter 6 covers the distribution theory of Fourier transforms (tempered distributions on R^1). Pseudo-measures are defined as distributions whose Fourier transforms are L^∞ functions. In a formal sense, pseudo-measures are the dual elements of $A(R)$. Their advantage over $L^\infty(R)$ is conceptual: e.g. if $\phi\in L^\infty(R)$, then the support of $\hat{\phi}$ in the distribution sense is equal to the "spectrum" of ϕ as defined, say, in Rudin's book. Chapter 7, on locally compact abelian groups, is a short resume, without proofs, of the basic facts.

Chapter 8, the last chapter, introduces Banach algebras, proves the Wiener-Levy theorems for $A(T)$ and $A(R)$, and develops (again in the classical T, R context) all of the "negative theorems" which round out the theory. These negative results, which say roughly that the only things which operate in $A(R)$ are the obvious ones, can be summarized:

(I) Every homomorphism (continuous or not) of the ring $A(R)$ into itself is given by an affine change of variable, $f(t)\rightarrow f(at+b)$, $a\neq 0$ (Beurling and Helson (1953)). In particular, if $\phi\colon R\rightarrow R$ is such that $f(\phi(t))\in A(R)$ for all $f\in A(R)$, then $\phi(t)=at+b$. [The problem of classifying the endomorphisms of $A(G)$ becomes much more difficult when G is not connected. The general $L\subset A$ case was finally settled by Paul Cohen in 1960.]

(II) If $\phi(t)$ is a complex function defined on $[-1, 1]$, and if for every real-valued $f\in A(T)$ with $-1\leqq f\leqq 1$, $\phi(f(t))$ also lies in $A(T)$, then $\phi(t)$ is real-analytic. (This is a converse to the Wiener-Levy theorem.) The results in this area are due to Helson, Kahane, Katznelson, and Rudin, and appeared in a number of papers published during 1958 and 1959.

Note. The following easy lemma practically reduces all problems concerning $A(R)$ to the corresponding problem for $A(T)$, and vice-versa:

A function f with support on $[\delta, 2\pi-\delta]$ belongs to $A(R)$ if and only if it belongs to $A(T)$.

(III) Nonspectral synthesis. The spectral synthesis hypothesis asserts that if $f, g\in A(G)$ (G a LCA group), and the zero-set of g contains that of f, then g lies in the closed ideal generated by f. It is false whenever G is not discrete (Malliavin (1959)). The author proves one case of Malliavin's theorem: for G equal to the "binary decimal group," $(Z_2)^\infty$ with the Tychonoff topology. This is probably not the case nearest to most reader's hearts! However, the construction is simpler than for $G=T$. [For $G=T$, one needs some lemma such as: As $N\to\infty$, the integral of the product $\int_0^{2\pi} f(t)g(Nt)dt$ approaches arbitrarily closely the product of the integrals $(\int_0^{2\pi} f)$ $(\int_0^{2\pi} g)$ (g being 2π-periodic, of course). For $G=(Z_2)^\infty$, Fubini's theorem does the whole trick.]

Thus it is remarkable that Malliavin's general theorem can be deduced from the "binary decimal" case. This is done by means of tensor products, an approach due to Varopoulos. What happens is that (G being given), a "very linearly independent" Cantor set $E\subset G$ is found, together with a many-to-one mapping ϕ of E onto $(Z_2)^\infty$, so that every non S (spectral synthesis) set in $(Z_2)^\infty$ is pulled back by ϕ onto a non S-set in G. Katznelson's book closes with an account of this recent and important theory.

J. IAN RICHARDS

An introduction to nonassociative algebras, by R. D. Schafer. Pure and Applied Mathematics, vol. 22, Academic Press, New York, 1966. x+166 pp. $7.95.

By ring or algebra is generally understood an associative ring or an associative algebra. This is a natural situation since, apart from a few books and expository articles dealing with a particular class of nonassociative algebras, this is the first book treating nonassociative algebras on a more general basis. These are algebras whose multiplication is not assumed to satisfy the associative law. It is easy to guess from our previous remark that a great part of the content of the book appears for the first time in book form, and those topics which have been already dealt with in other books are presented under a new light.

Besides organizing scattered material the author presents it in such a way that the reader can arrive at important theorems without