THE PSEUDO-CIRCLE IS NOT HOMOGENEOUS[1]

BY LAWRENCE FEARNLEY[2]

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1. **Introduction.** Two well-known and hitherto unsolved problems concerning limit spaces are those raised by R. H. Bing in 1951 in [2, p. 49]. These problems, which have also been discussed at the Summer Institute on Set Theoretic Topology of the American Mathematical Society, Madison, Wisconsin, 1955 by F. B. Jones [9], are the questions of whether or not all pseudo-circles are topologically equivalent and whether or not there exists a homogeneous pseudo-circle. In a recently completed paper [8], a research announcement of which was presented in [7], this author has given an affirmative solution to the first of these two problems. The purpose of this present paper is to give a complete solution to the second and more widely discussed problem.

2. **Preliminaries.** Throughout this paper we shall use terms and notations which are either the same as, or natural extensions of those developed by this author in [4], [5], [6] and [8]. In general these concepts were originally suggested by those used by Bing in [1], [2] and [3]. For convenience of reference we give statements of the definitions of the less standardized of these special terms that are needed in this paper. In the presentation of these definitions and in the development of the results of this paper it will be assumed that all circular chains have at least six links and the domains and ranges of all cyclic r-patterns have at least six elements. This will allow us to avoid the formal awkwardness of treating cases where the circular configurations and winding or crookedness characteristics are trivial, and will not involve any loss in generality in establishing the principal results of this paper.

The terms *r-pattern*, *cyclic r-pattern*, *winding number* and *linear representation* are defined in [5]. In this paper we make a minor extension of the definition of the first of these terms by allowing the domain of an r-pattern to be any set of consecutive integers, not necessarily a finite set. The r-pattern associated with a normal refine-

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ment [4] is called a *normal r-pattern.* A linear representation v_f of a cyclic r-pattern f will be said to be a *universal linear representation* of f if the domain of f is the set of all integers. If, in addition, $v_f(0)=0$, then v_f will be defined to be the *canonical universal linear representation of f.*

A cyclic r-pattern f is defined to be *left-normal at an index i* if, for any universal linear representation g of f, $g(i)$ is a minimum of the set $(g(i), g(i+1), g(i+2), \cdots)$. If $g(i)$ is a maximum of the set $(g(i), g(i-1), g(i-2), \cdots)$, then f is said to be *right-normal at the index i.* It is easily verified that if f is left-normal or right-normal at an index i with respect to any universal linear representation of f then f is left-normal or right-normal, respectively, at an index i with respect to all of its universal linear representations.

A *standard cyclic r-pattern f* is a cyclic r-pattern such that $f(0)=0$, f is left-normal at the index 0, and f has positive winding number.

A circular chain $P=P(0, n)$ is defined to be a *refinement* of a circular chain $Q=Q(0, m)$ if each link of P is contained in some link of Q. In this situation there is at least one cyclic r-pattern f such that a link of P with subscript i is contained in the link of Q with subscript $f(i)$, $i=0, 1, \cdots, n$. Then f is said to be a cyclic r-pattern of P in Q. The *winding number of P in Q* is the integer which is the winding number of every cyclic r-pattern of P in Q. The circular chain P is defined to be *a left-normal refinement of the circular chain Q at a point x* if there exists a cyclic r-pattern f of P in Q and an integer i such that f is left-normal at i and the link of P with subscript i contains the point x. The concept of a circular chain P being *a right-normal refinement of a circular chain Q at a point x* is defined in a similar manner.

If P is a circular chain which is a refinement of a circular chain Q and x is a point such that P is not a left-normal refinement of Q at x, then there is a nonnegative integer k with the property that, for any choice of a cyclic r-pattern f of P in Q, a universal linear representation v_f of f and a subscript z of a link of P containing x, there is an integer s such that $v_f(s)+k<v_f(z)$ and $s>z$. The greatest of all integers such as k is called the *left-displacement of P in Q at x.* The term *right-displacement of P in Q at x* is defined in a similar manner.

If $P=(p_a, p_{a+1}, \cdots, p_b)$ and $Q=(q_c, q_{c+1}, \cdots, q_d)$ are chains, the *ordered sum* of P and Q is the formal sequence of links $(p_a, p_{a+1}, \cdots, p_b, q_c, q_{c+1}, \cdots, q_d)$.

Finally, let $P=P(0, n)$ and $Q=Q(0, m)$ be circular chains and let f be a cyclic r-pattern of P in Q. A chain of links of P of the form $(p_i, p_{(i+1)\bmod n}, p_{(i+2)\bmod n}, \cdots, P_{(i+t)\bmod n})$, $t<n$, is said to have

f-winding number k in Q if $f(i)=f((i+t)\bmod n)$ and if, for any universal linear representation v_f of f, $v_f(i+t)-v_f(i)=k(m+1)$.

3. Compositions involving left-normality and right-normality. In this section we establish two preliminary results involving left-normality, right-normality, and patterns of the type described by Bing in his original definition of the pseudo-circle [2, p. 49]. These results will have important functions in the next section in giving a solution to the homogeneity problem for the pseudo-circle.

LEMMA 1. *Let $P=P(0, n)$, $Q=Q(0, m)$ and $R=R(0, t)$ be circular chains, let f be a standard cyclic r-pattern of P in Q with winding number 1, let g be a cyclic r-pattern of Q in R with winding number 1, and let P be equal to the ordered sum of chains $P(0, i)$, $P(i+1, j)$, $P(j+1, k)$ and $P(k+1, n)$ whose f-winding numbers are 4, -1, -1 and -1 respectively. In addition, let x be a point of some link of $P(j+1, k)$, let $f(i)=f(j)=f(k)$, and let the restriction $u_f|(j+1, j+2, \cdots, k)$ of the canonical universal linear representation u_f of f to the set $(j+1, j+2, \cdots, k)$ be a normal r-pattern. Then P is a refinement of R which has left-displacement at x of at least $t-3$ and has right-displacement at x of at least $t-3$.*

PROOF. First we choose universal linear representations v_g of g and v_{gf} of gf such that $v_{gf}=v_g u_f$. Then, since $u_f(j+1)=3(m+1)$, $u_f(k)=2(m+1)$ and $u_f|(j+1, j+2, \cdots, k)$ is a normal r-pattern, it follows that if d is the minimum of the range of $v_g|(0, 1, \cdots, m+1)$ then $d+2(t+1)$ is the minimum of the range of $v_g u_f|(i+1, j+2, \cdots, k)$. Next we choose an integer r, $k<r<n$, such that $u_f(r)=2(m+1)$ and $u_f|(r, r+1, \cdots, n)$ is a normal r-pattern. Then the minimum of the range of $v_g u_f|(r, r+1, \cdots, n)$ is $d+(t+1)$. Therefore, if z is the subscript of a link of P containing x there is an integer s such that $s>z$ and $v_g u_f(s)+(t-1)<v_g u_f(z)$. From this relationship and the fact that R is a circular chain we conclude that the refinement P of R has left-normal displacement at x of at least $t-3$.

A similar argument establishes the corresponding result involving the right-normal displacement of P in R at x.

LEMMA 2. *Let P, Q and R be circular chains such that P is a refinement of Q with winding number 1 in Q and Q is a refinement of R with winding number 1 in R. In addition, let x be a point of a link of P such that the refinement Q of R has left-displacement at x of at least 2 and has right-displacement at x of at least 2. Then the refinement P of R is neither left-normal nor right-normal at x.*

PROOF. We choose z to be the subscript of a link of P containing x, and let v_g and v_f be universal linear representations of f and g, respectively, such that $v_f(z)=f(z)$ and $v_g(f(z))=gf(z)$. Then, since the refinement Q of R has left-normal displacement at x of at least 2, there is an integer s such that $s>f(z)$ and $v_g(s)+2<v_g(f(z))$. Furthermore, since P has positive winding number in Q, there is an integer r such that $r>z$ and $v_f(r)=s$. Hence $v_gv_f(r)+2<v_gv_f(z)$, $r>z$. From this relationship and the fact that R is a circular chain it follows that the refinement P of R is not left-normal at x.

A similar argument proves that the refinement P of R is not right-normal at x.

4. Nonhomogeneity of pseudo-circles and generalized pseudo-circles. In this section we use the results of §3 in establishing that the pseudo-circle and all non-snake-like hereditarily indecomposable circularly chainable continua are nonhomogeneous.

THEOREM 1. *The pseudo-circle is not homogeneous.*

PROOF. Let M be a pseudo-circle considered as space, let x be a point of M, and note by [8] that M is topologically unique. Then there is a sequence of circular chains $\{T_i\}$ such that, for each positive integer j, T_j covers M and has mesh less than $1/j$, and the relationship of T_{j+1} to T_j at x is the same as the relationship described in Lemma 1 of P to Q at x. We may assume that there is a point p which is the intersection of the links of $\{T_i\}$ with subscripts zero.

Now suppose that there is a homeomorphism h of M onto M such that $h(p)=x$. Since h is uniformly continuous, there are integers r and s greater than 1 such that T_r is a refinement of $h(T_1)$ and $h(T_s)$ is a refinement of T_{r+1}. Hence, by Lemma 1, T_{r+1} has left-displacement and right-displacement in $h(T_1)$ at x of at least 2. Therefore, by Lemma 2, we obtain the contradiction to the choice of $\{T_i\}$ that $h(T_s)$ is neither left-normal nor right-normal in $h(T_1)$ at x.

From the same argument as that given above together with [6, Theorem 3.2] we also obtain the following additional result.

THEOREM 2. *There does not exist any non-snake-like hereditarily indecomposable circularly chainable continuum that is homogeneous.*

REMARK. The author is informed that J. T. Rogers, Jr. has also developed results related to those presented in this paper using different methods.

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BRIGHAM YOUNG UNIVERSITY, PROVO, UTAH 84601