ON SINGULARITIES OF SURFACES IN E^4

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1. **Notation.** Let $f\colon M^2 \to E^4$ be an immersion of a compact orientable surface. Let $e_1e_2e_3e_4$ be orthonormal righthanded frames, e_1e_2 tangent and agreeing with a fixed orientation of M. As usual define ω_i and ω_{ij} by

$$df = \sum \omega_i e_i \qquad de_i = \sum \omega_{ij} e_j, \quad i = 1, \cdots, 4.$$

The connection forms in the tangent and normal bundles are respectively ω_{12} and ω_{34}. The respective curvature forms are $d\omega_{12}$ and $d\omega_{34}$. The Gauss curvature K and the normal curvature N satisfy (and may be defined by)

$$d\omega_{12} = -K\omega_1 \wedge \omega_2, \qquad d\omega_{34} = -N\omega_1 \wedge \omega_2.$$

2. **Statement of the main results.**

THEOREM 1. *Suppose $f\colon M \to E^4$ is an immersion such that N is everywhere positive (negative). Then*

$$\chi(NM) = -2\chi(M) \qquad (\chi(NM) = 2\chi(M)).$$

Here $\chi(NM)$ is the Euler characteristic of the normal bundle and $\chi(M)$ is the Euler characteristic of M.

COROLLARY 2. *Every immersion of the sphere or torus must have a point where $N=0$.*

The proof of Theorem 1 uses a geometrically defined field of tangent axes. In order to define these axes we review some of the local theory of surfaces in E^4.

3. **The curvature ellipse** [1]. The local invariants of a surface in E^4 are characterized by an ellipse in the normal plane. To define this ellipse let us first define a map $\eta\colon S_p \to N_p$, S_p is the unit tangent circle at p and N_p is the normal plane at p. Let $\gamma(s)$ be a geodesic of M through p such that $d\gamma/ds(p) = e_1$, where e_1 is a unit vector at p. Define η by $\eta(e_1) = d^2\gamma/ds^2(p)$. The curvature ellipse is the image of S_p under η.

The mean curvature vector $\mathcal{H}$ is the position vector of the center of this ellipse.

4. **Construction of a field of axes** [1]. In general the line through the mean curvature vector meets the curvature ellipse in two diametrical points. The inverse image under η of these two points are four unit tangent vectors which form a pair of orthogonal tangent lines, i.e. an axis. This construction fails only when $\mathcal{H}=0$ or at an inflection point.

THEOREM 3. *The singular locus (inflection points and points where $\mathcal{H}=0$) of the field of axes constructed above is generically a set of isolated points. The index is generically* $\pm\frac{1}{2}$.

5. **Sketch of the proof of Theorem 1.** If $N>0$ then the point cannot be an inflection point. Thus the field of axes constructed above has singularities only at points where $\mathcal{H}=0$. These are generically isolated. Let them be $p_1 \cdots p_n$. Let $\mathrm{Ind}_1(p_i)$ be the index of this field of axes at p_i. Generically $\mathrm{Ind}_1(p_i)=\pm\frac{1}{2}$. On the other hand the mean curvature vector is a normal vector and hence gives a normal vector field with singularities at $p_1, \cdots, p_n$. Let $\mathrm{Ind}_2(p_i)$ be the index of $\mathcal{H}$ at p_i. Generically $\mathrm{Ind}_2(p_i)=\pm 1$. The proof then consists in showing that if $N>0$ then $\mathrm{Ind}_1(p_i)$ and $\mathrm{Ind}_2(p_i)$ have opposite signs (and if $N<0$ $\mathrm{Ind}_1(p_i)$ and $\mathrm{Ind}_2(p_i)$ have the same signs). Once this is established the proof follows readily from the fact that

$$\chi(NM) = \sum \mathrm{Ind}_2(p_i), \qquad \chi(M) = \sum \mathrm{Ind}_1(p_i).$$

6. **Proof of Corollary 2.**

$$\chi(NM) = \frac{1}{2\pi}\int_M N dA.$$

Thus if $N>0$ ($N<0$) everywhere then $\chi(NM)>0$ ($\chi(NM)<0$). By Theorem 1 if $N>0$ (or $N<0$) everywhere then $\chi(M)<0$. Consequently we obtain a contradiction if M is a torus or a sphere.

In the light of Theorem 1 it would be interesting to know of examples of immersions with everywhere positive N. We have not found any yet.

REFERENCES

1. C. L. E. Moore and E. B. Wilson, ***Differential geometry of two-dimensional surfaces in hyperspace***, Proc. Amer. Acad. Arts and Sci. 52 (1916), 267–368.

2. J. Little, ***On singularities of submanifolds of higher dimensional Euclidean spaces***, Thesis, University of Minnesota, Minneapolis, Minn., 1968.

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