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CLASSIFICATION OF KNOTS IN CODIMENSION TWO

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Introduction. In this paper we consider smooth knots, i.e., smooth embeddings $\phi: S^n \to S^{n+2}$, $n \geqq 3$. Two knots ϕ and η are said to be equivalent if there is a diffeomorphism $f: S^{n+2} \to S^{n+2}$ such that $f\phi(S^n) = \eta(S^n)$. The embedding ϕ extends to an embedding $\bar{\phi}: S^n \times D^2 \to S^{n+2}$, and any two such extensions are ambient isotopic relative to $S^n \times 0$. Hence if $A = \mathrm{cl}(S^{n+2} - \bar{\phi}(S^n \times D^2))$, the pair $(A, \partial A)$ is determined up to diffeomorphism by the equivalence class of ϕ. We call $(A, \partial A)$ the complementary pair, or simply the complement, of the knot ϕ. In this paper we show that if $\pi_1 A$, the fundamental group of the knot, is infinite cyclic, then there is at most one knot inequivalent to ϕ with complementary pair $(B, \partial B)$ of the same homotopy type as $(A, \partial A)$. This result is of interest because for any $n \geqq 3$ there are many inequivalent knots $\phi: S^n \to S^{n+2}$ with fundamental group $\boldsymbol{Z}$, see for example [12]. (The result also holds in the P.L. case, provided ϕ extends to a P.L.-embedding $\bar{\phi}: S^n \times D^2 \to S^{n+2}$.)

1. **Knots with diffeomorphic complements.** In [4], Gluck showed that homeomorphisms of $S^2 \times S^1$ are isotopic if and only if they are homotopic and used this result to conclude that there are at most two knots $\phi: S^2 \to S^4$ with homeomorphic exteriors. In [1], W. Browder studied the pseudo-isotopy classes of diffeomorphisms (and P.L. equivalences) of $S^1 \times S^n$ for $n \geqq 5$. He showed that two P.L. equivalences are pseudo-isotopic if and only if they are homotopic. For the group $\mathfrak{D}(S^n \times S^1)$ of pseudo-isotopy classes of diffeomorphisms, he obtained the exact sequence

$$\Upsilon^n + \Upsilon^{n+1} \to \mathfrak{D}(S^n \times S^1) \to \mathcal{E}(S^n \times S^1) \to 0,$$

where $\mathcal{E}(S^n \times S^1) = Z_2 + Z_2 + Z_2$ is the group of homotopy classes of homotopy equivalences of $S^1 \times S^n$ with itself. Using this result Browder, following Gluck, showed that there are at most two inequivalent knots $\phi: S^n \to S^{n+2}$, $n \geq 5$, with diffeomorphic complements. In this section we show that this result is also valid for $n = 3$ or 4.

PROPOSITION 1.1. *Let M^{n+1}, $n \geq 4$, be a P.L. manifold of the same homotopy type as $S^1 \times S^n$. Then M is P.L. homeomorphic to $S^1 \times S^n$.*

PROOF. For $n \geq 5$, it follows from the P.L. version of the main theorem of [2] or [3] that M^{n+1} is a P.L. bundle over S^1 with fiber of the homotopy type of S^n and hence P.L. equivalent to S^n. Therefore M can be obtained from $S^n \times I$ by identifying the two boundaries using an orientation preserving P.L. homeomorphism of S^n with itself. But such a P.L. homeomorphism is isotopic to the identity.

For $n = 4$, this proposition is just the P.L. version of a theorem of [8]. (See also Theorem 2.3 below.)

COROLLARY 1.2. *If $n = 4$, 5, any smooth M^{n+1} of the same homotopy type as $S^n \times S^1$ is diffeomorphic to $S^n \times S^1$.*

PROOF. In these dimensions P.L. manifolds have unique smooth structures, see [5].

THEOREM 1.3. *For $n = 3$, 4, any diffeomorphism d of $S^n \times S^1$ with itself which is homotopic to the identity is pseudo-isotopic to the identity.*

PROOF. Let $M^{n+2} = D^{n+1} \times S^1 \cup_d D^{n+1} \times S^1$. Since d is homotopic to the identity, M has the homotopy type of $S^{n+1} \times S^1$, and so is diffeomorphic to $S^{n+1} \times S^1$, by Corollary 1.2. Let $g: S^{n+1} \times S^1 \to M$ be a diffeomorphism. Writing $S^{n+1} \times S^1 = D^{n+1} \times S^1 \cup D^{n+1} \times S^1$, we may assume (by the tubular neighborhood theorem and a Whitney embedding theorem) that g carries the first summand in the decomposition of $S^{n+1} \times S^1$ into the first summand in the decomposition of M and that its restriction to these summands is a SO$(n+1)$-bundle map. Hence, after restricting to the second summands and composing with an SO$(n+1)$-bundle map, we get a diffeomorphism $h: D^{n+1} \times S^1 \to D^{n+1} \times S^1$ extending d; i.e., $h(x, y) = d(x, y)$ for x in ∂D^{n+1} and y in S^1. Let $D_0 = \frac{1}{2} D^{n+1}$ be the disk of radius $\frac{1}{2}$. Then by the tubular neighborhood theorem again, we can also insist that $h(D_0 \times S^1) = D_0 \times S^1$ and that $h | D_0 \times S^1$ is an SO$(n+1)$-bundle map. Hence d is pseudo-isotopic to a bundle map $\partial D_0 \times S^1 \to \partial D_0 \times S^1$, which represents $\beta \in \pi_1(\mathrm{SO}(n+1)) = Z_2$, say. Since d is homotopic to the identity, $\beta = 0$, (since the nontrivial element of $\pi_1(\mathrm{SO}(n+1))$ represents a nontrivial element of $\mathcal{E}(S^n \times S^1)$, by [1], for example). Hence d is pseudo-isotopic to the identity.

COROLLARY 1.4. *For $n=3, 4$, let $\mathfrak{D}(S^n \times S^1)$ be the group of pseudo-isotopy classes of diffeomorphisms of $S^n \times S^1$ into itself. Then the natural map $\mathfrak{D}(S^n \times S^1) \to \mathcal{E}(S^n \times S^1) \cong Z_2 + Z_2 + Z_2$ is an isomorphism.*

PROOF. By Theorem 1.3, it is monic, and each generator of $\mathcal{E}(S^n \times S^1)$ is represented by a diffeomorphism. (See [1].)

REMARK. For all $n \geqq 3$, arguments similar to the above can also be used to show that any homotopic P.L. homeomorphisms of $S^n \times S^1$ to itself are (P.L.) pseudo-isotopic.

The arguments of [1] can now be extended to lower dimensions by using Corollary 1.4. This yields the main result of this section.

THEOREM 1.5. *Let $n \geqq 3$. Then there are at most two inequivalent knots $\phi: S^n \to S^{n+2}$ with diffeomorphic complements.*

2. Knots with complements of the same homotopy type.

THEOREM 2.1. *Let $n \geqq 3$. Let $\phi_i: S^n \times D^2 \to S^{n+2}$, $i=1, 2$, be smooth (or P.L.) embeddings. Let $A_i = \mathrm{cl}(S^{n+2} - \mathrm{Im}\ \phi_i)$ and suppose that $\pi_1 A_i = Z$. Then if $(A_1, \partial A_1)$ and $(A_2, \partial A_2)$ have the same homotopy type (as pairs), A_1 and A_2 are diffeomorphic (resp. P.L. equivalent).*

COROLLARY 2.2. *If $\phi: S^n \to S^{n+2}$, $n \geqq 3$, is a smooth knot with fundamental group Z and complement $(A, \partial A)$, then there is at most one inequivalent knot with exterior $(B, \partial B)$ of the same homotopy type as $(A, \partial A)$,*

Corollary 2.2 follows immediately from Theorem 2.1. We recall that in case A has the homotopy type of a circle, ϕ is actually unknotted. (See [6] and [9].)

PROOF OF THEOREM 2.1. We concentrate on the smooth case. The P.L. case can be handled by similar methods. First assume $n \geqq 4$. By Alexander duality and the universal coefficient theorem, $H^j(A; G) = 0$ for $j \geqq 2$ and $H^j(A; G) = G$ for $j = 0, 1$, G any abelian group. Now let $h: (A_1, \partial A_1) \to (A_2, \partial A_2)$ be a homotopy equivalence of pairs, and let $\eta_h \in [A_2; F/O]$ be the "characteristic F/O-bundle" of h. (See [7] or [11].) F/O is connected and $\pi_1(F/O) = 0$, and so it follows from Theorem 3 of Chapter 8, §4 of [10], that $[A_2; F/O] = H^2(A_2; \pi_2(F/O)) = H^2(A_2; Z_2) = 0$. So $\eta_h = 0$. ($[A_2; F/O]$ = homotopy classes of maps of A_2 into F/O.) This means that there is a tangential cobordism of $(A_1, \partial A_1)$ with $(A_2; \partial A_2)$; i.e. there is a parallelizable W^{n+1} with

$\partial W = A_1 \cup \partial_0 W \cup A_2$, $\partial_0 W$ a cobordism of ∂A_1 with ∂A_2, and a retraction r: $(W, \partial W) \rightarrow (A_2, \partial A_2)$ such that $r|(A_1, \partial A_1)$ is homotopic to h. (See [7], [8] or [13].) Now, according to Wall [14] (see also 7.4 and 7.5 of [13]), one can perform surgery relative to $A_1 \cup A_2$ (i.e. without doing any modifications on $A_1 \cup A_2$) to get an s-cobordism. (This uses the fact that $\pi_1 A_2 = \mathbf{Z}$.) Thus we get an s-cobordism of $(A_1, \partial A_1)$ with $(A_2, \partial A_2)$, and so the relative s-cobordism theorem applies to prove 2.1 for $n \geqq 4$.

Now take $n = 3$. Then we have to use the following result from [8].

THEOREM 2.3. *Let M be obtained from S^5 by surgery on an embedded S^3. Then any manifold of the same homotopy type as M is diffeomorphic to M.*

Assuming Theorem 2.3, let h: $(A_1, \partial A_1) \rightarrow (A_2, \partial A_2)$ be a homotopy equivalence. Since every homotopy equivalence of $S^1 \times S^3$ with itself extends to a homotopy equivalence of $S^1 \times D^4$ with itself, it is easy to see that there is a homotopy equivalence k: $A_1 \cup_1^{\phi 0} D^4 \times S^1 \rightarrow A_2 \cup_{\phi_2} D^4 \times S$. Hence by Theorem 2.3, these manifolds are diffeomorphic. Using a Whitney theorem and the tubular neighborhood theorem again, it follows that there is a diffeomorphism of these manifolds which restricts to a diffeomorphism of A_1 with A_2. (Note that the S^1's in the second summands represent generators of the respective fundamental groups of these manifolds.)

REMARK. The above proof of 2.1 for $n \geqq 4$ is essentially a part of the proof that if $(M, \partial M)$ is a smooth manifold pair such that the inclusion induces an isomorphism of $\pi_1(\partial M)$ with $\pi_1 M$, then the "concordance classes of homotopy smoothings of $(M, \partial M)$" are in 1-1 correspondence with $[M; F/O]$ via the map induced by taking the "characteristic F/O-bundle" of a homotopy equivalence. This result is discussed in the simply-connected case in [7] and [11].

Note. Theorem 2.1, for $n \geqq 5$, was proved for fibred knots by W. Browder (*Manifolds with* $\pi_1 = Z$, Bull. Amer. Math. Soc. 72 (1966), 238–244, Corollary 2.4). Browder informs us that the requirement that the knots be fibred can be eliminated in his approach using recent results of Farrell-Hsiang.

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