STABLE MANIFOLDS FOR HYPERBOLIC SETS

BY MORRIS W. HIRSCH AND CHARLES C. PUGH

Communicated by Jürgen K. Moser, August 20, 1968

1. **Introduction.** We present a version of the "Generalized stable manifold theorem" of Smale [2, p. 781]. Details will appear in the Proceedings of the American Mathematical Society Summer Institute on Global Analysis.

Let M be a finite dimensional Riemannian manifold, $U \subset M$ an open set and $f: U \to M$ a C^k embedding ($k \in Z_+$). A set $\Lambda \subset U$ is a *hyperbolic set* provided

(1) $f(\Lambda) = \Lambda$;

(2) $T_\Lambda M$ has a splitting $E^s \oplus E^u$ preserved by Df;

(3) there exist numbers $C > 0$ and $\tau < 1$ such that for all $n \in Z_+$,

$$\max\{\|(Df \mid E^s)^n\|, \|(Df \mid E^u)^{-n}\|\} \leqq C\tau^n.$$

It is known (J. Mather; see also [1]) that the Riemannian metric on M can be chosen so that $C = 1$; we assume $C = 1$ in what follows. The splitting is unique.

Notation. If X is a metric space, $B_r(x) = \{y \in X \mid d(y, x) \leqq r\}$. If E is a Banach space, $BE = B_1(0)$. If $E \to X$ is a Banach bundle, $BE = \bigcup_{x \in X} BE_x$.

A submanifold $W \subset M$ is a *stable manifold through x of size β* if $W \cap B_\beta(x)$ is closed and consists of all $y \in B_\beta(x)$ such that $f^n(y)$ is defined and in $B_\beta f^n(x)$ for all $n \in Z_+$.

An *unstable manifold* is defined to be a stable manifold for f^{-1}. Unstable manifolds are easier to handle in proofs, but stable ones are easier to describe notationally. Hence, we confine ourselves to the stable case.

A C^k *stable manifold system with bundle* E is a family of C^k submanifolds $\{W_x\}_{x \in \Lambda}$ such that

(4) there exists $\beta > 0$ such that each W_x is a stable manifold through x of size β;

(5) E is a vector bundle over Λ, and there is a map $\phi: V \to M$ of a neighborhood V of the zero section of E such that ϕ maps each $V \cap E_x$ diffeomorphically onto W_x;

(6) ϕ is *fibrewise* C^k in this sense: Let $H: A \times R^q \to p^{-1}A$ be a trivialization of E over $A \subset \Lambda$ with $H(A \times D^q) \subset V$. Then each map $\theta_x = \phi \circ H \mid x \times D^q: D^q \to M$ is C^k, and $\theta: A \to C^k(D^q, M)$ is continuous.

2. Existence and uniqueness.

THEOREM 1. *Let* Λ *be a compact hyperbolic set for* $f: U \to M$. *Then there exists a* C^k *stable manifold system* $\{W_x\}_{x\in\Lambda}$ *with bundle* E^s *such that*

(a) *each* W_x *is tangent to* E^s_x *at* x;

(b) $(W_x - \partial W_x) \cap W_y$ *is an open (possibly empty) subset of* W_y *for all* x, y *in* Λ;

(c) *there exist numbers* $K>0$ *and* $\lambda<1$ *such that if* $x\in\Lambda$, $z\in W_x$ *and* $n\in Z_+$ *then* $d(f^n(x), f^n(z)) \leqq K\lambda^n$.

The proof is based on the following stable manifold theorem for a hyperbolic fixed point in a Banach space. The case $k=1$ is essentially contained in Chapter IX, Lemma 5.1 of Hartman [5].

Let $L(\cdot)$ denote Lipschitz constant.

THEOREM 2. *For* $i=0, 1$ *let* T_i *be an invertible linear operator on a Banach space* $\mathbf{E}_i$ *such that* $\max\{\|T_1^{-1}\|, \|T_0\|\} \leqq \tau < 1$. *There exists* $\epsilon>0$, *depending only on* τ, *with the following properties. Put* $\mathbf{E} = \mathbf{E}_0 \times \mathbf{E}_1$ *and* $T = T_0 \times T_1$.

(a) *If* $f: B\mathbf{E} \to \mathbf{E}$ *satisfies* $\max\{|f(0)|, L(f-T)\} < \epsilon$, *there exists a unique map* $g: B\mathbf{E}_0 \to B\mathbf{E}_1$ *such that*

$$\text{graph } g = B\mathbf{E} \cap f^{-1}(\text{graph } g);$$

(b) $x \in \text{graph } g$ *if and only if* $f^n(x) \in B\mathbf{E}$ *for all* $n \geqq 0$;

(c) $L(g) \leqq 1$; *and* g *is* C^k *if* f *is* C^k, *and* g *depends* C^k *continuously on* f.

Such a T is a *hyperbolic* linear map. We call g a *stable manifold function* for f.

OUTLINE OF PROOF OF THEOREM 1. Let S^b denote the Banach space $S^b(T_\Lambda M)$ of bounded sections of $T_\Lambda M$, and $S^c \subset S^b$ the closed subspace of continuous sections. Let $B^b = BS^b$ and $B^c = BS^c$ denote the unit balls. For $x\in\Lambda$ let $e_x: M_x \to M$ be the exponential map. If the metric on M is multiplied by a large constant there will be a C^k map $f_b: B^b \to S^b$ given by the formula

$$f_b(\sigma)f(x) = \overset{-1}{e_{fx}} f e_x \sigma(x).$$

(We motivate the definition of f_b by considering the Banach manifold $\mathfrak{M}(\Lambda, M)$ of bounded maps $\Lambda \to M$ and the local diffeomorphism

$$F: \mathfrak{M}(\Lambda, U) \to \mathfrak{M}(\Lambda, M)$$

given by

$$F(h) = f \circ h \circ (f^{-1} \mid \Lambda).$$

A coordinate chart for $\mathfrak{M}(\Lambda, M)$ with values in $S^b(T_\Lambda M)$ is obtained by letting the section σ correspond to the map $x \mapsto e_x\sigma(x)$ of Λ into M. The expression for F in these coordinates is then f_b.) This is the natural action of f on sections σ.

The derivative of f_b at 0 is the hyperbolic linear map $Df_b(0)\sigma = Df \circ \sigma \circ f^{-1}$; the corresponding splitting of S^b is $S^b(E^s) \oplus S^b(E^u) = S^b_s \oplus S^b_u$. We may assume $L(f_b - Df_b(0))$ so small that by Theorem 1, f_b has a C^k stable manifold function G^b: $B^b_s \to B^b_u$ (where $B^b_s = BS^b_s$, etc.). Similarly let G^c: $B^c_s \to B^c_u$ be the stable manifold function of $f_c = f_b | B^c$: $B^c \to S^c$.

LEMMA. *If $x_0 \in \Lambda$ and $\sigma_1, \sigma_2 \in B^b_s$ are such that $\sigma_1(x_0) = \sigma_2(x_0)$ then $G^b(\sigma_1)x_0 = G^b(\sigma_2)x_0$.*

PROOF. $G^b(\zeta)$ is the unique section ξ such that $|f^n_b(\zeta x, \xi x)| \leqq 1$ for all $n \in Z_+$ and $x \in \Lambda$, by Theorem 2(b). Applying this to

$$\begin{aligned} \zeta_i(x) &= 0 && \text{if } x \neq x_0, \\ &= \sigma_i(x_0) && \text{if } x = x_0 \end{aligned}$$

proves the lemma.

The lemma implies that $G^c = B^b | B^c_s$, and that there is a function H: $BE^s \to BE^u$ such that $G^b(\sigma) = H \circ \sigma$. Also $G^c(\sigma) = H \circ \sigma$, implying the continuity of H. Each map H_x: $BE^s_x \to BE^u_x$ is C^k. The map ϕ: $BE^s \to M$ is defined by $\phi(y) = e(y, H(y))$. It can be shown that ϕ is fibrewise C^k by writing ϕ as the composition.

$$BE^s \xrightarrow{(\chi, p)} B^c_s \times \Lambda \xrightarrow{G^c \times 1} B^c_u \times \Lambda \xrightarrow{v} M$$

where p is the bundle projection of E^s; χ: $BE^s \to B^c_s$ is a fibrewise C^k map assigning to each $y \in BE^s$ a section of E^s through y of norm $\leqq 1$; and v is the evaluation map $v(\sigma, x) = \sigma(x)$.

3. Smoothness of the splitting of $T_\Lambda M$.

THEOREM 3. *Let f: $U \to M$ be C^2, and suppose Λ is a compact hyperbolic set which is a C^2 submanifold. Then E^s is a C^1 subbundle of $T_\Lambda M$ provided $\|Df | E^u\| \cdot \|Df^{-1} | E^u\| \cdot \|Df | E^s\| < 1$. In particular this holds if E^s has codimension 1.*

The special case $\Lambda = M$ gives

COROLLARY 4. *Let f be a C^2 Anosov diffeomorphism of a compact manifold M. If the stable manifolds have codimension 1 they form a C^1 foliation of M.*

This was stated for dim $M=2$ in Anosov [4]. On the other hand, Arnold and Avez [3] state that if E^s has *dimension* 1 then the stable manifolds form a C^1 foliation.

OUTLINE OF PROOF OF THEOREM 3. Give $T_\Lambda M$ a C^1 splitting $F^s \oplus F^u$ approximating $E^s \oplus E^u$. For each $x \in \Lambda$ the subspace $E_x^s \subset M_x$ is the graph of a linear map $G_x \colon F_x^s \to F_x^u$, and

$$\text{graph } G_{fx} = Df(x)(\text{graph } G_x).$$

We consider G_x as an element in the bundle $L \to \Lambda$ whose fibre over x is the Banach space L_x of linear maps $F_x^s \to F_x^u$; then G is a section of L invariant under the map $\Gamma \colon BL \to BL$ defined as follows. Write $Df^{-1} \colon F^s \oplus F^u \to F^s \oplus F^u$ as a matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \colon F^s \to F^s$, $B \colon F^u \to F^s$, $C \colon F^s \to F^u$, and $D \colon F^u \to F^u$ are maps covering $f^{-1} \colon \Lambda \to \Lambda$. Define $\Gamma_x \colon L_x \to L_{f^{-1}x}$ by

$$\Gamma_x(\lambda) = (C_x + D_x\lambda) \circ (A_x + B_x\lambda)^{-1}.$$

Theorem 3 is proved once we know that G is C^1. This follows from

THEOREM 5. *Let $E \to M$ be a C^1 Banach bundle. Let $h \colon M \to M$ be a diffeomorphism covered by a C^1 map $\Gamma \colon BE \to BE$. Let $\alpha < 1$ be such that each map $\Gamma_x \colon BE_x \to BE_{hx}$ has Lipschitz constant $\leqq \alpha$. Then BE has a unique section σ invariant under Γ. Moreover σ is C^1 provided $\|Dh^{-1}\| < \alpha^{-1}$.*

REFERENCES

1. R. B. Holmes, *A formula for the spectral radius of an operator*, Amer. Math. Monthly **75** (1968), 163–166

2. S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.

3. V. I. Arnold and A. Avez, *Problèmes ergodiques de la mécanique classique*, Gauthier-Villars, Paris, 1967.

4. D. V. Anosov, *Roughness of geodesic flows on closed Riemannian manifolds of negative curvature*, Dokl. Akad. Nauk. SSSR **145** (1962), 707–709 = Soviet Math. Dokl. **3** (1962), 1068–1069.

5. P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.

UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720