ASYMPTOTIC PROPERTIES OF ENTIRE FUNCTIONS EXTREMAL FOR THE cos $\pi\rho$ THEOREM

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Let $f(z)$ be an entire function of order $\rho<1$. The classical "cos $\pi\rho$ theorem" of Valiron and Wiman [4, pp. 40, 51] asserts that if

$$\mu(r) = \min_{|z|=r} |f(z)|, \qquad M(r) = \max_{|z|=r} |f(z)|,$$

then, given $\epsilon>0$, the inequality

$$\log \mu(r) > (\cos \pi\rho - \epsilon)\log M(r) \tag{1}$$

holds for a sequence $r=r_n\to+\infty$.

We consider those functions $f(z)$ for which (1) is the best possible inequality, and discuss the global asymptotic behavior of such functions.

THEOREM 1. *Let $f(z)$ be an entire function of order ρ $(0\leqq\rho<1)$, and suppose*

$$\log \mu(r) \leqq [\cos \pi\rho + \epsilon(r)] \log M(r) \tag{2}$$

where $\epsilon(r)\to 0$ as $r\to\infty$.

Then there exists a set E of logarithmic density zero and a slowly varying function[2] *$\psi(r)$ such that*

$$\log M(r) = r^{\rho}\psi(r) \qquad (r \notin E), \tag{3}$$

$$n(r, 0) = [\sin \pi\rho/\pi + o(1)]r^{\rho}\psi(r) \qquad (r\to\infty, r\notin E) \tag{4}$$

(where, as usual, $n(r, 0)$ denotes the number of zeros of $f(z)$ in $|z|\leqq r$),

$$\log \mu(r) = [\cos \pi\rho + o(1)]r^{\rho}\psi(r) \qquad (r\to\infty, r\notin E\cup H), \tag{5}$$

where H has (linear) density zero.

Further, there exists a real-valued function $\theta(r)$ such that if $k>1$ and $\delta>0$ are given and $\nu(r)$ denotes the number of zeros of $f(z)$ in the region

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[2] A function $\psi(r)$ is said to vary slowly if it is defined and positive for all $r>r_0$ and satisfies $\lim_{r\to\infty} \psi(\sigma r)/\psi(r)\to 1$ $(0<\sigma<\infty)$. For a useful discussion of the properties of such functions see, for example, [9, p. 419]. For a discussion of linear and logarithmic densities see [4, p. 5].

$$\{z: k^{-1}r \leqq |z| \leqq kr, \delta \leqq |\arg z - \theta(r)| \leqq \pi\},$$

then

$$\nu(r) = o(r^{\rho}\psi(r)) \qquad (r \to \infty, r \notin E). \tag{6}$$

The function $\theta(r)$ oscillates slowly outside of E, in the sense that if $k>1$ and $\epsilon>0$ are given, then

$$|\theta(t) - \theta(r)| < \epsilon \qquad (r > r_0(\epsilon, k), r \notin E) \tag{7}$$

holds for all t in the interval $k^{-1}r \leqq t \leqq kr$.

The content of the conclusions (3)–(7) can be expressed more intuitively if we say that on almost all long intervals, $f(z)$ behaves like a Lindelöf function of order ρ [12, p. 18]. Indeed, it is not difficult to see that Theorem 1 implies that the asymptotic expansion

$$\log |f(te^{i(\phi+\phi_0)})| = [\cos \phi\rho + o(1)]\psi(r)t^{\rho} \qquad (k^{-1}r \leqq t \leqq kr, t \notin H, |\phi| \leqq \pi), \tag{8}$$

is valid (uniformly in t and ϕ) as $r \to \infty$ outside E, where $\phi_0 = \theta(r) - \pi$, $k>1$ is a given constant, and H is the set of density zero given in Theorem 1.

Recent examples[3] of W. K. Hayman [10] show that some exceptional set E must be present in Theorem 1; when coupled with Theorem 2 below, they also show that even in the important special case when all the zeros of $f(z)$ are negative, E cannot be replaced by a set of linear density 0.

Theorem 1 may be compared to recent results of Kjellberg [11], Essén [7], Essén-Ganelius [8], and Anderson [1]. These authors consider (2) from another point of view; in particular, ρ can be any number, $0<\rho<1$ (not necessarily the order of $f(z)$), but on the other hand, $\epsilon(r)$ must satisfy some condition such as

$$\limsup_{R_1, R_2 \to \infty} \int_{R_1}^{R_2} \frac{\epsilon(r) \log M(r)}{r^{1+\rho}} dr < M < \infty.$$

Their conclusion, that $\log M(r)/r^{\rho}$ tends to a limit α $(0 \leqq \alpha \leqq \infty)$ as $r \to \infty$ (with no need to avoid an exceptional set), is also of a different nature than that deduced here.

1. Outline of the proof. Let $f(z)$ satisfy the hypotheses of the theorem. We can assume that $f(0) = 1$, and write

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[3] Hayman's examples are valid only if $\rho = 1/2$, but he comments that this case is probably typical.

$$(1.1)\qquad f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right), \qquad F(z) = \prod_{n=1}^{\infty}\left(1 + \frac{z}{|a_n|}\right).$$

Since

$$(1.2)\qquad \log |F(-r)| + \log F(r) \leqq \log \mu(r) + \log M(r)$$

[4, p. 40], it follows at once that $F(z)$ also satisfies the hypotheses of Theorem 1, and hence a theorem of P. D. Barry [3] yields that

$$\log |F(-r)| = [\cos \pi\rho + o(1)] \log F(r) \qquad (r \to \infty, r \in G^*),$$

where G^* has logarithmic density one. It is not hard to see, using (1.2) and hypothesis (2) again, that

$$(1.3)\qquad \log M(r) \sim \log F(r) \qquad (r \to \infty, r \in G^*).$$

An easy extension of Theorem 2 of [2] now shows that from (1.3) follows

$$(1.4)\qquad \nu(r) = o(\log M(r)) \qquad (r \to \infty, r \in G^*).$$

We next establish that G^* can be replaced by a subset G_* having the following crucial properties: there are sequences $\{\alpha_n\}$, $\{\beta_n\}$ and a set H of (linear) density zero such that

$$G_* = \bigcup_{n=1}^{\infty} [\alpha_n, \beta_n] - H \qquad (\alpha_n \to \infty, \beta_n/\alpha_n \to \infty)$$

has logarithmic density one, and, if $k > 1$ is given, then

$$[k^{-1}\alpha_n, k\beta_n] \subset G^* \cup H \qquad (n > n_0(k)).$$

The exceptional set E which appears in the statement of Theorem 1 is the complement of $G = \bigcup_{n=1}^{\infty} [\alpha_n, \beta_n]$.

In view of elementary properties of sets of linear density zero, it is easy to see that (1.3) holds with G^* replaced by G, and so it suffices to prove (3)–(5) for $F(z)$. The argument hinges now on a suitable generalization (to allow exceptional sets of logarithmic density zero) of the following theorem [6], which is one form of a complement to some classical results of Titchmarsh [13] and Bowen and Macintyre [5].

THEOREM 2. *Let $F(z)$ be an entire function of the form* (1.1), *and suppose*

$$\frac{\log |F(-r)|}{\log F(r)} \to \alpha \qquad (r \to \infty, r \notin H),$$

where H is of (linear) density zero.

Then $-1 \leqq \alpha \leqq 1$, *and*

$$\log F(r) = r^{\rho}\psi(r),$$

$$n(r,0) = \left[\frac{\sin \pi\rho}{\pi} + o(1)\right] r^{\rho}\psi(r) \qquad (r \to \infty),$$

$$\log |F(-r)| = [\cos \pi\rho + o(1)] r^{\rho}\psi(r) \qquad (r \to \infty,\ r \notin H),$$

where ρ *is determined by*

$$\cos \pi\rho = \alpha \qquad (0 \leqq \rho \leqq 1),$$

and ψ *is a slowly varying function.*

Finally, (6) follows from (1.4) and (3), and (7) is an easy consequence of (6) (cf. [2, Corollary 1]).

Conclusion (3) of Theorem 1 implies that $f(z)$ has regular growth in the sense of Valiron. Analogues of Theorem 1, valid for functions of irregular growth, can also be derived from these methods.

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