AN ALGEBRA OF SINGULAR INTEGRAL OPERATORS WITH TWO SYMBOL HOMOMORPHISMS

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1. Let $R_+^{n+1} = \{(x, y) = (x_1, \cdots, x_n, y) : x_j, y \in R, y \geqq 0\}$ and let Δ_d, Δ_n denote the two unbounded positive self-adjoint operators of the Hilbert-space $\mathfrak{H} = \mathfrak{L}^2(R_+^{n+1})$ generated by closing the Laplace operator in $C_0^\infty(R_+^{n+1})$ under Dirichlet and Neumann boundary conditions at $y=0$, respectively.

We propose to study the "convolution algebra" $\mathfrak{A}^\#$ generated by the generalized Riesz-Hilbert-operators

$$\begin{aligned} &\Lambda_d = (1 - \Delta_d)^{-1/2}, \quad \Lambda_n = (1 - \Delta_n)^{-1/2}, \quad S_d = - i\partial/\partial y \Lambda_d, \\ (1) \quad &S_n = - i\partial/\partial y \Lambda_n, \quad S_{d,j} = - i\partial/\partial x_j \Lambda_d, \quad S_{n,j} = - i\partial/\partial x_j \Lambda_n, \\ &\qquad j = 1, \cdots, n \end{aligned}$$

and later on also will adjoin certain multiplications by continuous functions, to obtain an algebra $\mathfrak{A}$ of singular integral operators on the half-space R_+^{n+1}.

Both C^*-algebras $\mathfrak{A}^\#$ and $\mathfrak{A}$ have noncompact commutators, but each is commutative modulo a certain larger ideal ($\mathfrak{E}^\#$ and $\mathfrak{E}$, respectively). We therefore obtain a first symbol function σ_A for $A \in \mathfrak{A}^\#$ (or $\mathfrak{A}$) which is a continuous complex-valued function over the maximal ideal space of $\mathfrak{A}^\#/\mathfrak{E}^\#$ (or $\mathfrak{A}/\mathfrak{E}$). If σ_A does not vanish, we can invert the operator mod $\mathfrak{E}^\#$ (or $\mathfrak{E}$), or reduce the singular integral equation $A_n u = f$ to an equation $(1+E)u = g$ with $E \in \mathfrak{E}^\#$ (or $\mathfrak{E}$).

Now, we find that the ideals $\mathfrak{E}^\#$ and $\mathfrak{E}$ are isomorphic to topological tensor products of the form $\mathfrak{C}(\mathfrak{h}) \hat{\otimes} \mathfrak{S}^\#$, $\mathfrak{E} = \mathfrak{C}(\mathfrak{h}) \hat{\otimes} \mathfrak{S}$, with respect to a suitable direct decomposition

$$\mathfrak{H} = \mathfrak{h} \otimes \mathfrak{k}, \qquad \mathfrak{h} = \mathfrak{L}^2(R^+), \qquad \mathfrak{k} = \mathfrak{L}^2(R^n),$$

where $\mathfrak{C}(\mathfrak{h})$ denotes the compact ideal of $\mathfrak{h}$, while $\mathfrak{S}^\#$ and $\mathfrak{S}$ are certain algebras of singular integral operators over the boundary R^{n+1}.

Therefore to each operator $E \in \mathfrak{E}^\#$ (or $\mathfrak{E}$) there can be associated an operator valued symbol $\tau_E(m) \in \mathfrak{C}(\mathfrak{h})$ such that $1+E$ is Fredholm if and only if $1+\tau_E(m)$ is regular for all m. The construction of a Fredholm inverse for $A \in \mathfrak{A}$ will therefore depend on two symbols: first we invert the operator modulo $\mathfrak{E}$, if the complex-valued symbol

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σ_A does not vanish; then we invert an operator $1+E$ mod $\mathfrak{C}$, which depends on another, operator-valued symbol.

2. It is well known that the operators (1) have representations as (regular or singular) integral operators. Specifically

$$\Lambda_d = \Lambda_- - \Lambda_+, \qquad \Lambda_n = \Lambda_- + \Lambda_+$$

with

$$\Lambda_\pm u = (2/\pi)^{1/2}(2\pi)^{-(n+1)/2}\int_{R_+^{n+1}} K_{n/2}(t_\pm)t_\pm^{-n/2}u(x',y')dx'dy' \tag{2}$$

and

$$t_\pm = (|x-x'|^2+|y\pm y'|^2)^{1/2}, \tag{3}$$

where $K_\nu(s)$ denotes the modified Bessel function as in Magnus-Oberhettinger [6, p. 28]. All other operators (1) experience similar decompositions and we therefore may generate $\mathfrak{A}^\#$ by the following operators as well, which are integral operators:

$$\Lambda_\pm, S_\pm = -i\partial/\partial y\Lambda_\pm, \quad S_{j,\pm} = -i\partial/\partial x_j\Lambda_\pm, \qquad j=1,\cdots,n. \tag{4}$$

Note that

$$\begin{aligned} S_\pm u &= i(2/\pi)^{1/2}(2\pi)^{-(n+1)/2} \\ &\quad \cdot \int_{R_+^{n+1}} K_{n/2+1}(t_\pm)(y\pm y')/t_\pm^{n/2+1}u(x',y')dx'dy' \end{aligned} \tag{5}$$

and

$$\begin{aligned} S_{j,\pm} u &= i(2/\pi)^{1/2}(2\pi)^{-(n+1)/2} \\ &\quad \cdot \int_{R_+^{n+1}} K_{n/2+1}(t_\pm)(x_j-x_j')/t_\pm^{n/2+1}u(x',y')dx'dy'. \end{aligned} \tag{6}$$

3. Let F denote the unitary operator of $\mathfrak{H}$ induced by the Fourier transform, with respect to the x-variable only:

$$Fu(x,y) = (2\pi)^{-n/2}\int_{R^n} e^{-ix\cdot\xi}u(\xi,y)d\xi \quad \text{for } u\in C_0^\infty(R_+^{n+1}) \tag{7}$$

and let the unitary operator T be defined by

$$(Tu)(x,y) = \sigma^{-1/2}u(x,y/\sigma), \qquad (T^{-1}u)(x,y) = \sigma^{1/2}u(x,y\sigma) \tag{8}$$

with $\sigma=(1+|x|^2)^{1/2}$. Let $U=TF$; then we find that

(9) $US_{\pm}U^{-1} = P_{\pm}, \quad US_{j,\pm}U^{-1} = x_j/\sigma Q_{\pm}, \quad U\Lambda_{\pm}U^{-1} = 1/\sigma Q_{\pm}$

with

$$P_{\pm}u = i/\pi \int_0^{\infty} K_1(|y \pm y'|) \operatorname{sgn}(y \pm y')u(x, y')dy' \tag{10}$$

and

$$Q_{\pm}u = 1/\pi \int_0^{\infty} K_0(|y \pm y'|)u(x, y')dy', \tag{11}$$

with sgn $t=0$, ± 1 if $t=0$, >0, <0, resp.

Let $\mathfrak{h}=\mathfrak{L}^2(R^1_+)$ and, for a moment, let D_d and D_n denote the operators Δ_d and Δ_n as introduced initially, but for $n=0$. Then we see at once that we may reinterpret $P_{\pm}$, $Q_{\pm}$ above as operators on $\mathfrak{h}$ and that then

$$\begin{aligned} &(1 - D_d)^{-1/2} = Q_- - Q_+, && (1 - D_n)^{-1/2} = Q_- + Q_+, \\ &-i\partial/\partial y(1-D_d)^{-1/2}=P_- - P_+, && -i\partial/\partial y(1-D_n)^{-1/2}=P_- + P_+, \end{aligned} \tag{12}$$

while we get

$$\mathfrak{H} = \mathfrak{k} \hat{\otimes} \mathfrak{h}, \qquad \mathfrak{k} = \mathfrak{L}^2(R^n) \tag{13}$$

and the relations (9) take the form

$$\begin{aligned} US_{\pm}U^{-1} = I \otimes P_{\pm}, \qquad US_{j,\pm}U^{-1} = (x_j/\sigma) \otimes Q_{\pm}, \\ U\Lambda_{\pm}U^{-1} = (1/\sigma) \otimes Q_{\pm}. \end{aligned} \tag{14}$$

In (13) $\hat{\otimes}$ denotes the topological tensor product.

4. We notice that the operators $P_{\pm}$, $Q_{\pm}$ of §1 are evidently in the algebra $\mathfrak{F}$ as introduced in [4, §5]. In particular, Q_+ is a compact operator of $\mathfrak{h}$, Q_- is an even Wiener-Hopf convolution with $\mathfrak{L}^1$-kernel, and $P_{\pm}$ differ from $cK^0_{\pm}$, with the operators $K^0_{\pm}$ as in [4] and a suitable constant c, only by a compact operator each. It is also easily seen that $\mathfrak{F}$ may be generated as a C^*-algebra with unit by $\mathfrak{C}(\mathfrak{h})$, $P_{\pm}$, $Q_{\pm}$ as well as by the generators listed in [4].

DEFINITION. (a) $\mathfrak{G}^\#$ denotes the C^*-subalgebra of $\mathfrak{L}(\mathfrak{H})$ without unit generated by the operators of the form

$$U^*(a(x) \otimes C)U \tag{15}$$

with $a(x)\in \mathcal{C}(B^n)$, $C\in\mathfrak{C}(\mathfrak{h})$.

(b) $\mathfrak{A}^{\#}$ denotes the C^*-algebra with unit generated by $\mathfrak{G}^{\#}$ above and all operators $S_{\pm}$, $S_{j,\pm}$, $\Lambda_{\pm}$, $j=1, \cdots, n$.

Note. As in [5] B^n denotes the smallest compactification of R^n into which the mapping $\rho: R^n \to \{|x|<1\}$ defined by

$$(16) \quad \rho(x) = (2/\pi)x/|x| \arctan |x|, \quad x \neq 0, \qquad =0 \quad \text{for } x = 0$$

can be continuously extended.

We then have

THEOREM 1. *$\mathfrak{G}^{\#}$ is a closed two-sided ideal of the C^*-algebra $\mathfrak{A}^{\#}$. The algebra $\mathfrak{A}^{\#}/\mathfrak{G}^{\#}$ is commutative and isometrically isomorphic to the function algebra $\mathcal{C}(\mathfrak{M}^{\#})$ with the compact Hausdorff space $\mathfrak{M}^{\#}$ obtained from the product $B^n \times \mathfrak{M}(\mathfrak{F})$ by identifying all points of B^n over each point of the straight line segment $x=0$, $-\infty < t < +\infty$, $\xi = \infty$ in the space $\mathfrak{M}(\mathfrak{F})$ as defined in* [4, §5].

Clearly $\mathfrak{G}^{\#}$ does not contain compact operators, except 0. On the other hand, $\mathfrak{G}^{\#}$ is contained in the R-algebra $U^*(\mathfrak{C}(\mathfrak{k}) \hat{\otimes} \mathfrak{L}(\mathfrak{h}))U = \mathfrak{J}$ and Theorem 1 relates the $\mathfrak{J}$-Fredholm property of $A \in \mathfrak{A}^{\#}$ to the non-vanishing of a continuous function over $\mathfrak{M}^{\#}$. (See [2], [3].)

Note that $M^{\#}$ is homeomorphic to an $n+1$-ball B^{n+1} with the endpoints of a one-dimensional interval I^1 attached to it at two distinguished points.

5. Let H^{n+1} denote the closure of R_+^{n+1} in B^{n+1}. It then is an easy consequence of results published in [5] that the commutators $[S_{\pm}, b]$, $[S_{j,\pm}, b]$, $[\Lambda_{\pm}, b]$, $j=1, \cdots, n$ are all in $\mathfrak{C}(\mathfrak{H})$, for $b \in \mathcal{C}(H^{n+1})$.

DEFINITION. (a) $\mathfrak{E}$ denotes the C^*-algebra *without* unit generated by $\mathfrak{C}(\mathfrak{H})$ and all products bE, Eb, $b \in C(H^{n+1})$, $E \in \mathfrak{G}^{\#}$.

(b) $\mathfrak{A}$ denotes the C^*-algebra *with* unit generated by $\mathfrak{C}(\mathfrak{H})$, $\mathfrak{A}^{\#}$ and $C(H^{n+1})$.

We then have the following main result.

THEOREM 2. (a) *$\mathfrak{E} \subset \mathfrak{A}$ is a closed two-sided ideal of $\mathfrak{A}$, and $\mathfrak{A}/\mathfrak{E}$ is commutative.*

(b) *$\mathfrak{E} = \mathfrak{C}(\mathfrak{H})$ is a closed two-sided ideal of E.*

(c) *The Gelfand space $\mathfrak{M}$ of $\mathfrak{A}/\mathfrak{E}$ is (homeomorphic to) the following subset of the cartesian product $\mathfrak{M}^{\#} \times H^{n+1}$ ($\mathfrak{M}^{\#}$ as in Theorem 1):*

(i) *Over the boundary at $y=\infty$ of H^{n+1} one gets all points of $B^{n+1} \subset \mathfrak{M}^{\#}$.*

(ii) *Over interior points of $R_+^{n+1} \subset H^{n+1}$ one gets the boundary ∂B^{n+1} of the ball $B^{n+1} \subset \mathfrak{M}^{\#}$.*

(iii) *Over the boundary $y=0$ of $R_+^{n+1} \subset H^{n+1}$ one gets the interval I^1 and the boundary ∂B^{n+1} of the ball $B^{n+1} \subset H^{n+1}$.*

(iv) *Over the points* $y=0$, $|x| = \infty$ *of* $\boldsymbol{H}^{n+1}$ *one gets the whole space* $\mathfrak{M}^{\#}$.

(d) *The algebra* $\mathfrak{E}/\mathfrak{C}$ *is isometrically isomorphic to the algebra* $\mathcal{C}(\mathfrak{M}_1, \mathfrak{C}(\mathfrak{h}))$ *of all continuous functions from a compact Hausdorff-space* $\mathfrak{M}_1$ *to the compact ideal* $\mathfrak{C}(\mathfrak{h})$ *of the Hilbert-space* $\mathfrak{h}$.

(e) *The space* $\mathfrak{M}_1$ *is (homeomorphic to) the set*

$$\partial B^n \times B^n \cup B^n \times \partial B^n \subset B^n \times B^n, \tag{17}$$

(i.e., topologically is a $2n-1$ *sphere).*

DEFINITION. (a) To any $A \in \mathfrak{A}$ we associate $\sigma_A \in \mathcal{C}(\mathfrak{M})$ defined as the function associated to the coset of A mod $\mathfrak{E}$ by the Gelfand isomorphism of $\mathfrak{A}/\mathfrak{E}$. σ_A will be called the $\mathfrak{E}$-symbol of $A \in \mathfrak{A}$.

(b) To any $E \in \mathfrak{E}$ we associate $\tau_E \in \mathcal{C}(\mathfrak{M}_1, \mathfrak{C}(\mathfrak{h}))$ defined as image of the coset of E mod $\mathfrak{C}(\mathfrak{H})$ under the isomorphism (d) of Theorem 2. τ_E will be called the $\mathfrak{C}$-symbol of $E \in \mathfrak{E}$.

THEOREM 3. (a) *A necessary condition for* $A \in \mathfrak{A}$ *to be Fredholm is that its* $\mathfrak{E}$*-symbol does never vanish on* $\mathfrak{M}$.

(b) $A \in \mathfrak{A}$ *with* $\sigma_A \neq 0$ *on* $\mathfrak{M}$ *possesses an inverse* $B \in \mathfrak{A}$ mod $\mathfrak{E}$ *such that* $1-AB$, $1-BA \in \mathfrak{E}$.

(c) $A \in \mathfrak{A}$ *with* $\sigma_A \neq 0$ *is Fredholm if and only if for some* $\mathfrak{E}$*-inverse* B *of* A *we have*

$$(1 + \tau_{(AB-1)}(m)) \text{ a regular operator of } \mathfrak{L}(\mathfrak{h})$$

for all $m \in \mathfrak{M}_1$.

6. The proof of Theorem 2 rests on the following facts partly of independent interest.

THEOREM 4. *If* $b \in C(\boldsymbol{H}^{n+1})$ *vanishes on the boundary* $y=0$ *then* Eb, bE *are compact, for all* $E \in \mathfrak{E}^{\#}$.

The result of Theorem 4 may be expressed by saying that all operators of $\mathfrak{E}$ are "compact, except over the boundary of $\boldsymbol{R}^{n+1}$."

THEOREM 5. *We have* $U\mathfrak{E}U^* = \mathfrak{S}\hat{\otimes}\mathfrak{C}(\mathfrak{h})$ *with the algebra* $\mathfrak{S}$ *as in* [5, *appendix*].

This completely clarifies the structure of the ideal $\mathfrak{E}$ and assertions (d) and (e) of Theorem 2 become evident, in view of [5] and [1].

While the proof of assertions (a) and (b) is a verification only, one may employ techniques as in [5] to obtain the precise extent of the space $\mathfrak{M}$.

We notice that the operators of our algebra $\mathfrak{A}$ are similar to those considered by Vishik and Eskin [7], for instance.

Applicability of our results should strongly depend on the explicit construction of inverses mod $\mathfrak{C}$ and of Fredholm inverses.

Especially we also expect results concerning pseudo-differential operators involving boundary conditions in a half-space like those in [**4**, §6].

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