COHOMOLOGY OF CERTAIN STEINBERG GROUPS

BY DAVID HERTZIG

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In [3] Steinberg considers the relations satisfied by generators of the Chevalley groups and defines certain abstract groups Δ and Γ via presentation. Let Σ be a root system of a simple complex Lie algebra $\mathfrak{G}_C$ and let K be a field of characteristic $p \geqq 0$. We consider a set of generators $x_r(t)$ $(r \in \Sigma,\ t \in K)$ and the relations

(A) $x_r(t)x_r(u) = x_r(t+u) \quad (r \in \Sigma;\ t, u \in K),$

(B) $x_r(t)x_s(u)x_r(t)^{-1} = x_s(u)\prod x_{ir+js}(C_{ij;rs}t^iu^j)$

$$(r, s \in \Sigma,\ r + s \neq 0;\ t, u \in K).$$

The product in (B) is over all integers $i, j \geqq 1$ for which $ir+js \in \Sigma$, taken in lexicographic order. The $C_{ij;rs}$ are certain integers depending only on the structure of $\mathfrak{G}_C$ (cf. [1]). Steinberg defines $w_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t)$ and $h_r(t) = w_r(t)w_r(-1)$ $(r \in \Sigma;\ t \in K^*)$ and considers also the relations

(B′) $w_r(t)x_r(u)w_r(t^{-1}) = x_{-r}(-t^{-2}u) \quad (r \in \Sigma;\ t \in K^*,\ u \in K),$

(C) $h_r(t)h_r(u) = h_r(tu) \quad (r \in \Sigma;\ t, u \in K^*).$

The *Steinberg group* Δ is the abstract group generated by the symbols $x_r(t)$ $(r \in \Sigma;\ t \in K)$ subject to the relations (A) and (B) if the rank of Σ is >1, to the relations (A) and (B′) if the rank of $\Sigma = 1$. The *Steinberg group* Γ is the abstract group with the same generators as Δ subject to the relations of Δ and in addition subject to the relations (C).

In [1] Chevalley constructs a corresponding Lie algebra $\mathfrak{G}$ over the field K and there is a natural action of the Steinberg groups on $\mathfrak{G}$. One is then led to consideration of the cohomology $H^1(\Delta, \mathfrak{G})$ and $H^1(\Gamma, \mathfrak{G})$.

The author has developed a technique for computation of such cohomology (cf. [2]). This is applied successfully to obtain the following results. Proofs will appear elsewhere.

We denote $\mathfrak{D}(K)$ the module of derivations of K. In the case of characteristic $p=2$ we denote $\mathfrak{L}(K)$ the K^2-linear transformations L of K such that $L(1)=0$. Since $p=2$, $\mathfrak{D}(K) \subset \mathfrak{L}(K)$, but in general $\mathfrak{D}(K) \neq \mathfrak{L}(K)$.

THEOREM 1. $H^1(\Delta, \mathfrak{G}) = H^1(\Gamma, \mathfrak{G}) \cong \mathfrak{D}(K)$ *in the following cases*:

(i) *type* A_1, $p \neq 2$ *and* $K \neq F_5$;

(ii) *type* A_n $(n \geqq 2)$, $p \nmid n+1$;
(iii) *type* D_n $(n \geqq 4)$, $p \neq 2$;
(iv) *type* E_6, $p \neq 3$;
(v) *type* E_7, $p \neq 2$;
(vi) *type* E_8;
(vii) *type* F_4.

THEOREM 2. $H^1(\Delta, \mathfrak{G}) = H^1(\Gamma, \mathfrak{G}) \cong K \oplus \mathfrak{D}(K)$ *in the following cases*:
(i) *type* A_n $(n \geqq 2)$, $p \mid n+1$;
(ii) *type* D_n $(n \geqq 4$, *n odd*$)$, $p = 2$;
(iii) *type* E_6, $p = 3$;
(iv) *type* E_7, $p = 2$.

THEOREM 3. $H^1(\Delta, \mathfrak{G}) = H^1(\Gamma, \mathfrak{G}) \cong K \oplus K \oplus \mathfrak{D}(K)$ *in the case*: *type* D_n $(n \geqq 4$, *n even*$)$, $p = 2$.

THEOREM 4. $H^1(\Delta, \mathfrak{G}) = H^1(\Gamma, \mathfrak{G}) \cong K$ *in the case*: *type* A_1, $K = \mathbf{F}_5$.

THEOREM 5. $H^1(\Delta, \mathfrak{G}) \cong K \oplus \mathfrak{L}(K)$, $H^1(\Gamma, \mathfrak{G}) \cong K \oplus \mathfrak{D}(K)$ *in the case*: *type* A_1, $p = 2$.

REFERENCES

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3. R. Steinberg, *Générateurs, relations et revêtements de groupes algébriques*, Colloque Theory des Groupes Algebriques, Brussels Libraire Universitaire Louvain and Gauthier-Villars, Paris 1962, pp. 113–127.

PURDUE UNIVERSITY, LAFAYETTE, INDIANA 47907