THE CONVEX HULL OF THE FINITE BLASCHKE PRODUCTS[1]

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Let U be the unit disc in the complex plane, $U = \{z \mid |z| < 1\}$. A finite Blaschke product is an analytic function on U which is either a constant of modulus one or of the form

$$(*) \qquad \lambda \prod_{i=1}^{N} \frac{z - \alpha_i}{1 - \bar{\alpha}_i z}, \qquad N = 1, 2, \cdots$$

where λ is a complex number of modulus one and $\alpha_i \in U$ for $1 \leqq i \leqq N$. It follows from [1, p. 12] that the set of functions analytic on U and continuous on $\bar{U}$, the closure of U, which have modulus one on the unit circle consists precisely of the finite Blaschke products. The purpose of this note is to prove the following theorem, originally raised as a question by Phelps in [2].

THEOREM. *Let f be analytic on U, continuous on $\bar{U}$, and bounded by one. Then f may be uniformly approximated on $\bar{U}$ by convex combinations of finite Blaschke products.*

PROOF. For $0 \leqq t \leqq 1$ let $f_t(z) = f(tz)$ for $z \in U$. Since f is continuous on $\bar{U}$, f_t converges uniformly to f as $t \to 1$. By a theorem of Caratheodory [1, p. 13] there is a sequence $\{\Phi_j\}$ of finite Blaschke products of the form (*) such that $\Phi_j \to f$ uniformly on compact subsets of U. It follows that given $\epsilon > 0$ and $t < 1$ we may find a finite Blaschke product Φ of the form (*) with $\|f_t - \Phi_t\| < \epsilon/2$. Thus given $\epsilon > 0$ there is a $t < 1$ and a finite Blaschke product Φ of the form (*) with $\|f - \Phi_t\|_\infty < \epsilon$. We now show that Φ_t is itself actually a convex combination of finite Blaschke products. It suffices to do this when $\Phi(z) = (z - \alpha)(1 - \bar{\alpha} z)^{-1}$ since $(gh)_t = g_t h_t$ for any g and h and since the set of convex combinations of finite Blaschke products is clearly closed under multiplication. However, for $\alpha = re^{i\theta}$ and $0 \leqq t \leqq 1$ we have

$$\frac{tz - \alpha}{1 - \bar{\alpha} tz} = \frac{t(1 - r^2)}{1 - r^2 t^2} \left\{ \frac{z - \alpha t}{1 - \bar{\alpha} tz} \right\} + \frac{r(1 - t^2)}{1 - r^2 t^2} \{-e^{i\theta}\} + \frac{(1 - t)(1 - r)}{1 + rt} \{0\}.$$

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Since the factors in the first two sets of brackets are finite Blaschke products and the zero in the third is a convex combination of such, and since the coefficients are nonnegative and sum to 1, the proof is complete.

REFERENCES

1. C. Caratheodory, *Theory of functions*. Vol. 2, Chelsea, New York, 1954.
2. R. R. Phelps, *Extreme points in functions algebras*, Duke Math. J. **32** (1965), 267–278.

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EXACTNESS OF INVERSE LIMITS

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I. The problem of this investigation is to characterize those small categories X for which the inverse limit

$$\operatorname{proj}\lim_{X}: AB^X \to AB$$

is exact. Here AB is the category of abelian groups, and AB^X is the category of functors from X to AB. In this context I conjecture the following

THEOREM I. *Let X be a small category. Then the following assertions are equivalent*:

(1) *The inverse limit* proj $\lim_X$: $AB^X \to AB$ *is exact.*

(2) *For every abelian category $\mathfrak{A}$ with exact direct products, the inverse limit* proj $\lim_X$: $\mathfrak{A}^X \to \mathfrak{A}$ *is exact.*

(3) *Every connected component Y of X contains an object y together with an endomorphism $e \in Y(y, y)$ such that the following properties are satisfied*:

(i) *y is a smallest object of Y, i.e., for any object $z \in Y$ there is a morphism $y \to z$.*

(ii) *e equalizes any two morphisms with the same codomain and domain y, i.e., any diagram $y \xrightarrow{e} y \rightrightarrows z$ is commutative.*

At present, I can prove the equivalence of (1) and (2) and the implication (3)$\rightrightarrows$(1) in general, i.e., without any additional condition on X. The implication (1)$\rightrightarrows$(3) holds at least if one of the following conditions on X is satisfied: