THE COHOMOLOGICAL DIMENSION OF STONE SPACES

BY ROGER WIEGAND

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The purpose of this note is to announce a few inequalities involving the cohomological (sheaf-theoretic) dimension of locally compact, totally disconnected Hausdorff spaces, herein called *Stone spaces*. Throughout, R will denote a commutative regular ring with maximal ideal space X. (Then X is compact and totally disconnected.) For each ideal J in R let $U[J]$ denote the corresponding open subset of X, and for each R-module A, let $\mathcal{A}(A)$ denote the corresponding sheaf of modules, as defined in [2].

THEOREM 1. $\operatorname{Ext}_R^n(J, A)$ *and* $H^n(U[J]; \mathcal{A}(A))$ *are naturally isomorphic.*

THEOREM 2. *Let* $\mathfrak{F}$ *be a sheaf over the Stone space* X, *and let* $\mathfrak{U}$ *be a covering of* X *consisting of compact open sets. Then the natural maps* $H^n(\mathfrak{U}; \mathfrak{F}) \to \check{H}^n(X; \mathfrak{F}) \to H^n(X; \mathfrak{F})$ *are all isomorphisms.*

Let dim X denote the cohomological dimension of X, and cov dim X the covering dimension of X, based on arbitrary (not necessarily finite) open coverings. (It is not hard to show that for Stone spaces, cov dim $X \leqq n$ iff X has a compact open cover of order n.) Finally, let $h\cdot\dim_R J$ denote the homological (projective) dimension of the ideal J.

COROLLARY. $h\cdot\dim_R J \leqq \dim U[J] \leqq \text{cov} \dim U[J]$.

Since the only projective R-modules are direct sums of principal ideals [1], we see that $h\cdot\dim_R J = 0$ iff cov dim $U[J] = 0$, and, by the corollary, iff dim $U[J] = 0$. In order to see that equality need not always hold in the corollary, let us define the *rank* ρ of a space X by agreeing that $\rho(X) \leqq n$ iff X can be written as a union of $\aleph_n$ (or fewer) compact sets.

THEOREM 3. *For any Stone space* X, $\dim X \leqq \rho(X)$.

EXAMPLE 1. Let Ω be the set of countable ordinals, with the order topology. Then $\dim \Omega = 1$, but cov dim $\Omega = \infty$. (The second assertion may be verified directly; the first then follows from Theorem 3 and the remarks following the corollary.)

The next example shows that the inequality in Theorem 3 cannot be sharpened.

EXAMPLE 2. For each $n \geqq 0$, let X_n be the product of $\aleph_n$ copies of a two point space, with a single point deleted. Then $\dim X_n = \rho(X_n) = n$. (Pierce [3] has shown that the corresponding maximal ideal in the free Boolean ring on $\aleph_n$ generators has homological dimension n. Therefore $\dim X_n = n$, by Theorem 3 and the corollary.)

EXAMPLE 3. Let A_0 (resp. A_1) be the one-point compactification of a discrete space of cardinality $\aleph_0$ (resp. $\aleph_1$). Let $X = A_0 \times A_1 - \{(*, *)\}$. Then $\dim X = \text{cov dim } X = \rho(X) = 1$. (In fact, it can be shown that $H^1(X; Z_2) \neq 0$, where Z_2 denotes the constant 2-sheaf.)

I do not know whether the identity $\dim X = \text{cov dim } X = n$ can be realized in general. (The space X_n of Example 2 has infinite covering dimension.) An obvious generalization of Example 3 yields a space with rank and covering dimension n, but with unknown cohomological dimension. Also, I know of no example in which $h \cdot \dim_R J < \dim U[J]$. Notice that if one could show that $h \cdot \dim_R J$ and $\dim U[J]$ are always equal, then it would follow that any two commutative regular rings with homeomorphic maximal ideal spaces have the same global dimension.

REFERENCES

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2. R. S. Pierce, *Modules over commutative regular rings*, Mem. Amer. Math. Soc. No. 70 (1967).

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UNIVERSITY OF WISCONSIN