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NONEXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS OF NONLINEAR EIGENVALUE PROBLEMS[1]

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We consider nonlinear eigenvalue problems of the general form:

$$Lu = F(\lambda, x, u), \qquad x \in D, \tag{1}$$

$$\beta(x)\partial u/\partial\nu + \alpha(x)u = 0, \qquad x \in \partial D. \tag{2}$$

Here $x = (x_1, x_2, \cdots, x_m)$ and

$$\left.\begin{aligned} &L\phi \equiv \sum_{i,j=1}^{m} \partial_i[a_{ij}(x)\partial_j\phi] - a_0(x)\phi, \qquad a_{ij}(x) = a_{ji}(x) \\ &\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \geqq a^2 \sum_{i=1}^{m} \xi_i^2, \qquad a^2 > 0; \qquad a_0(x) \geqq 0 \end{aligned}\right\} x \in D;$$

$$\left.\begin{aligned} &\frac{\partial\phi}{\partial\nu} \equiv \sum_{i,j=1}^{m} n_i(x)a_{ij}(x)\partial_j\phi \\ &\alpha(x)\beta(x) \geqq 0, \qquad \alpha(x) \not\equiv 0, \qquad \alpha(x) + \beta(x) > 0 \end{aligned}\right\} x \in \partial D. \tag{3}$$

All coefficients and the derivatives of the $a_{ij}(x)$ are continuous on the appropriate closed sets $\bar{D}$ or ∂D, and the latter is piecewise smooth with exterior unit normal vector $(n_1(x), n_2(x), \cdots, n_m(x))$ at $x \in \partial D$. We first prove a simple but useful result on conditions for the nonexistence of positive solutions of (1)–(2).

THEOREM 1. *Let $F(\lambda, x, z)$ be continuous on $x \in D$, $z > 0$. For any positive continuous function $r(x)$ on $\bar{D}$, let $\mu_1\{r\}$ be the least eigenvalue of*

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$$(4)\qquad \begin{aligned} L\psi + \mu r(x)\psi &= 0, \qquad x \in D, \\ \beta(x)\partial\psi/\partial\nu + \alpha(x)\psi &= 0, \qquad x \in \partial D. \end{aligned}$$

Then (1)–(2) *has no positive solution for any* $\lambda \in \Lambda\{r\}$ *where*

$$(5)\quad \Lambda\{r\} \equiv \{\lambda \mid F(\lambda, x, z) + \mu_1\{r\}r(x)z \neq 0, \qquad \text{all } x \in D, z > 0\}.$$

PROOF. Suppose (1)–(2) has a positive solution, $u(x) > 0$, $x \in D$, for a given fixed λ. Then this solution trivially satisfies

$$Lu + \mu_1\{r\}r(x)u = F(\lambda, x, u) + \mu_1\{r\}r(x)u$$

and (2). Since L is selfadjoint, the right-hand side must be orthogonal to $\psi_1(x)$, the eigenfunction of (4) belonging to $\mu_1\{r\}$. From (3) it follows that $\psi_1(x)$ is of one sign on D. Thus the orthogonality relation requires that the continuous right-hand side change sign on D. Hence $\lambda \notin \Lambda\{r\}$. ■.

Of course piecewise continuous $F(\lambda, x, u)$ and $r(x) > 0$ are easily included by replacing $\neq 0$ in definition (5) by either alternative: >0 or <0. The above theorem generalizes some nonexistence results contained in Keller & Cohen [1].

We now consider some special cases of (1)–(2) in which positive solutions are known or conjectured to exist. The problems are of the form:

$$(6)\qquad \begin{aligned} Lu + \lambda r(x)u &= f(x, u), \qquad x \in D, \\ \beta(x)\partial u/\partial\nu + \alpha(x)u &= 0, \qquad x \in \partial D, \end{aligned}$$

where $r(x)$ is continuous and positive on D.

Some nonexistence results for the above problem are a simple consequence of Theorem 1.

COROLLARY 1.1. (a) *For some constant k let $f(x, z) > kr(x)z$ for all $z > 0$ and $x \in D$. Then* (6) *has no positive solutions for any* $\lambda \leq \mu_1\{r\} + k$.

(b) *For some constant k let $f(x, z) < kr(x)z$ for all $z > 0$, and $x \in D$. Then* (6) *has no positive solution for any* $\lambda \geq \mu_1\{r\} + k$.

PROOF. (a) $F(\lambda, x, z) \equiv f(x, z) - \lambda r(x)z > (k - \lambda)r(x)z \geq -\mu_1\{r\}r(x)z$ for $z > 0$ if $\lambda \leq \mu_1\{r\} + k$. Then $\lambda \in \Lambda\{r\}$.

(b) As above, we see that $F(\lambda, x, z) < -\mu_1\{r\}r(x)z$ if $\lambda \geq \mu_1\{r\} + k$.

Note that k in the Corollary may have either sign, but the case $k = 0$ is of particular interest. It implies that if (6) is to have positive solutions for all $\lambda \geq 0$, then $f(x, z)$ must change sign on $z > 0$, $x \in D$. In a recent paper D. S. Cohen [2] proves that (6) has unique positive solutions for $0 \leq \lambda < \mu_1\{r\}$ when $f(x, z) \equiv -f(x) + g(x, z)$ where: $f(x) < 0$, $g(x, z) > 0$, $g_z(x, z) > 0$, $g_{zz}(x, z) > 0$ and $g_z(x, z)z > g(x, z)$ for all $z>0$, $x\in D$. It can be shown that if $f(x, 0)<0$, $f_z(x, z)>0$ for all $z>0$, $x\in D$ and $\lim_{z\to\infty} f_z(x, z)=\infty$, then (6) has positive solutions for all λ. Under these conditions D. Cohen has observed that a result of Levinson [3] implies that (6), with $L\equiv\Delta$ and $\beta\equiv 0$, has *solutions* for all values of λ. We shall show that positive solutions of (6) are unique if only $f_z(x, z)$ is increasing in z for $z>0$ and $f(x, 0)\leqq 0$.

THEOREM 2. *Let $f(x, z)$ have a continuous z-derivative and satisfy for all $x\in D$:*

$$\text{(7)}\qquad \begin{aligned} &\text{(a) } f(x, 0) \equiv f_0(x) \leqq 0, \\ &\text{(b) } f_z(x, z) > f_z(x, z') > 0 \qquad \text{if } z > z' > 0. \end{aligned}$$

Then positive solutions of (6) *are unique* (*for all λ for which they exist*).

PROOF. Assume $u(x)$ and $v(x)$ are distinct positive solutions of (6) for the same value of λ. Then since $f_z(x, z)$ is continuous for $z>0$, we have

$$f(x, u(x)) - f(x, v(x)) = q(x, u(x), v(x))[u(x) - v(x)]$$

where

$$\text{(8)}\qquad q(x; u, v) \equiv \int_0^1 f_z(x, tu(x) + (1 - t)v(x))dt.$$

Thus with $w(x)\equiv u(x)-v(x)$, we obtain from (6) for u and v:

$$\text{(9)}\qquad \begin{aligned} &Lw + [\lambda r(x) - q(x; u, v)]w = 0, \qquad x \in D, \\ &\beta(x)\partial w/\partial\nu + \alpha(x)w = 0, \qquad x \in \partial D. \end{aligned}$$

Noting that $f(x, u(x))-f(x, 0)=q(x; u(x), 0)u(x)$ we can write (6) as

$$\text{(10)}\qquad \begin{aligned} &Lu + [\lambda r(x) - q(x; u, 0)]u = f_0(x), \qquad x \in D, \\ &\beta(x)\partial u/\partial\nu + \alpha(x)u = 0, \qquad x \in \partial D. \end{aligned}$$

Now consider the two eigenvalue problems, with eigenvalue parameters σ and τ:

$$\text{(11a)}\qquad \begin{aligned} L\phi + [\sigma r(x) - q(x; u, v)]\phi &= 0, \qquad x \in D, \\ \beta(x)\partial\phi/\partial\nu + \alpha(x)\phi &= 0, \qquad x \in \partial D; \end{aligned}$$

$$\text{(11b)}\qquad \begin{aligned} L\psi + [\tau r(x) - q(x; u, 0)]\psi &= 0, \qquad x \in D, \\ \beta(x)\partial\psi/\partial\nu + \alpha(x)\psi &= 0, \qquad x \in \partial D. \end{aligned}$$

The least eigenvalue, σ_1 and τ_1 respectively, of each of these problems can be characterized by the variational principle:

$$\sigma_1 = \min_{\phi \in \mathfrak{A}} \left\{ Q[\phi] + \int_D \int q(x; u, v)\phi^2(x)dx \right\} \Big/ H[\phi],$$

$$\tau_1 = \min_{\phi \in \mathfrak{A}} \left\{ Q[\phi] + \int_D \int q(x; u, 0)\phi^2(x)dx \right\} \Big/ H[\phi].$$

Here the class of admissable functions is, say, $\mathfrak{A} \equiv \{\phi | \phi \in C(\bar{D}) \cap C'(D);\ \phi(x) = 0,\ x \in \partial D_1\}$ where $\beta(x) = 0$ if and only if $x \in \partial D_1$, $\partial D = \partial D_1 \cup \partial D_2$, $\partial D_1 \cap \partial D_2 = 0$ and:

$$Q[\phi] \equiv \int_D \int \left[\sum_{i,j=1}^{m} a_{ij}(x)\partial_i\phi\partial_j\phi + a_0(x)\phi^2 \right] dx + \int_{\partial D_2} \frac{\alpha(x)}{\beta(x)} \phi^2 ds,$$

$$H[\phi] \equiv \int_D \int r(x)\phi^2 dx.$$

Since $f_z(x, z)$ is increasing in z for $z > 0$ and $v(x) > 0$ on D, we must have for all $x \in D$,

$$q(x; u(x), v(x)) > q(x; u(x), 0).$$

Thus from the above variational principle it follows that

$$\sigma_1 > \tau_1.$$

By assumption, $w(x) \not\equiv 0$, and so the parameter λ appearing in (9) must be some eigenvalue of the problem (11a). Since σ_1 is the least eigenvalue of that problem we must have $\lambda \geq \sigma_1 > \tau_1$. Now write (10) as:

$$Lu + [\tau_1 r(x) - q(x; u, 0)]u = f_0(x) + (\tau_1 - \lambda)r(x)u(x), \qquad x \in D,$$
$$\beta(x)\partial u/\partial \nu + \alpha(x)u = 0, \qquad x \in \partial D.$$

But τ_1 is the least eigenvalue of (11b) and so the right-hand side in the above differential equation must be orthogonal to $\psi_1(x)$, the eigenfunction belonging to τ_1. However, this is impossible since $\psi_1(x)$ is of one sign on D and, since $u(x)$ is a positive solution,

$$f_0(x) + (\tau_1 - \lambda)r(x)u(x) < 0 \qquad \text{on } D.$$

The contradiction implies $w(x) \equiv 0$. ∎

The above proof remains valid if we relax the monotonicity condition (7b) to just nondecreasing derivative, $f_z(x, z) \geq f_z(x, z')$, $z > z' > 0$; but strengthen condition (7a) to $f_0(x) < 0$. Clearly our result also applies to the case with $f \equiv f(\lambda, x, u)$ provided (7) holds for the appropriate values of λ.

Many additional results have been obtained under the hypothesis of Theorem 2; namely: (i) positive solutions of (6) are increasing functions of λ for all $x \in D$; (ii) the set of λ for which positive solutions of (6) exist is open above; (iii) if $f_0(x) \equiv 0$, then (6) has no positive solutions for all $\lambda \leqq \lambda_1$ where λ_1 is the least eigenvalue of

$$L\phi + [\lambda r(x) - f_u(x; 0)]\phi = 0, \qquad x \in D,$$
$$\beta(x)\partial\phi/\partial\nu + \alpha(x)\phi = 0, \qquad x \in \partial D;$$

(iv) if $f_0(x) < 0$ on D, then (6) has positive solutions for all $\lambda < \lambda_1$; (v) if $f_0(x) < 0$ on D and a positive solution of (6) exists for some λ', then positive solutions exist for all $\lambda \leqq \lambda'$.

Also, we can show that (6) has a positive solution for arbitrarily large λ if in addition to (7) and $f_0(x) < 0$ on D we have $\lim_{z \to \infty} f_z(x, z) = +\infty$ on D. Combined with (v) above and Theorem 2 this yields unique positive solutions of (6) for all λ. The results in (i)–(v) are proven by combining the technique in Theorem 2 with the use of the Positivity Lemma as in [1], and are thus constructive results. Variational procedures are employed to show existence for arbitrarily large λ. The detailed proofs will be given elsewhere.

References

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