PLANARITY IN ALGEBRAIC SYSTEMS

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Planarity was introduced into algebra by Marshall Hall in his well-known coordinatization of a projective plane by a planar ternary ring [4]. In [6], J. L. Zemmer defines a near-field to be planar when the equation $ax = bx + c$ has a unique solution for $a \neq b$. In our investigation of planarity, we discovered that if $(N, +, \cdot)$ is a near-ring satisfying the above equational property, then $(N, +, \cdot)$ is a near-field. (This was conjectured by both D. R. Hughes and J. L. Zemmer in private communications.) We present some extensions of this result together with geometric interpretations of "planar" near-rings.

Definitions and notations. By a *left distributive system* is meant a triple $(N, +, \cdot)$ such that multiplication $\cdot$ is left distributive over addition $+$. Elements $a, b \in N$ are called *left equivalent multipliers*, denoted by $a \equiv_m b$ *iff* $ax = bx$ for all $x \in N$. The relation $\equiv_m$ is *discrete* when $a \equiv_m b$ implies $a = b$. A left distributive system is said to possess the *planar property* if the equation $ax = bx + c$ has a unique solution for $a \not\equiv_m b$.

DEFINITION. A left distributive system $(N, +, \cdot)$ with planar property is a *planar system* if

(1) in $(N, +)$ the right cancellation law is valid;

(2) in $(N, +)$ there is an identity 0;

(3) $(N, \cdot)$ is a semi-group;

(4) there are at least three points in N, no two of which are left equivalent multipliers.

A planar system is *integral* if 0 is the only left zero divisor.

Main results. Let $(N, +, \cdot)$ be an integral planar system. Then $0 \cdot x = x \cdot 0 = 0$ for all $x \in N$. Let 1_a be the solution to the equation $a \cdot x = a$, $a \neq 0$, and $B_a = \{x \in N^* \mid x \cdot 1_a = x\}$, where N^* denotes the nonzero elements of N. We have the following

THEOREM 1. *Let $(N, +, \cdot)$ be an integral planar system. Then*

(i) *each $(B_a, \cdot)$ is a group with identity 1_a;*

(ii) *the family $\{B_a\}_{a \in N^*}$ is pairwise disjoint;*

(iii) $N^* = \bigcup_{a \in N^*} B_a$;

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(iv) *$N^*B_a = B_a$ for each $a \in N^*$;*

(v) *if $a, c \in N^*$, then $\phi: B_a \to B_c$ defined by $\phi(x) = x1_c$ is an isomorphism;*

(vi) *each 1_a is a left identity for $(N, +, \cdot)$.*

COROLLARY. *Let $(N, +, \cdot)$ be a near-ring that is an integral planar system with $\equiv_m$ discrete. Then $(N, +, \cdot)$ is a planar near-field.*

PROOF. If $a, b \in N^*$, *then* $1_a \equiv_m 1_b$.

In the sequel a near-ring that is an integral planar system will be called an *integral planar near-ring.*

THEOREM 2. *Suppose $(N, +, \cdot)$ is an integral planar near-ring and each $\overline{B}_a = \{0\} \cup B_a$ is an additive normal subgroup. Also suppose that no $\overline{B}_a = N$ but any two $\overline{B}_a$, $\overline{B}_c$ generate N under $+$. Then*

(i) *each $(\overline{B}_a, +, \cdot)$ is a near-field;*

(ii) *$(\overline{B}_a, +, \cdot)$ is isomorphic to $(\overline{B}_c, +, \cdot)$ if $(x+y)1_c = x1_c + y1_c$ for all $x, y \in B_a$;*

(iii) *$(N, +)$ is abelian and is isomorphic to the direct sum $\overline{B}_a \oplus \overline{B}_c$ as groups;*

(iv) *the points of N are the points of an affine plane A with the cosets of the $\overline{B}_a$ as lines;*

(v) *the plane A can be coordinatized by a skew field.*

PROOF. The group $(N, +)$ is a Φ(I, IV) group [5]. A Φ(I, IV) group is abelian since $x \to x+g$ induces a translation on A and so Axiom 4a is satisfied (p. 58 of [1]). Axiom 4bP (p. 63 of [1]) holds at $0 \in N$ where $x \to tx$ are the required dilatations.

THEOREM 3. *Suppose $(N, +, \cdot)$ is a finite integral planar near-ring and each $\overline{B}_a = \{0\} \cup B_a$ is an additive subgroup. Also suppose that no $\overline{B}_a = N$ but any two $\overline{B}_a$, $\overline{B}_c$ generate N under $+$. Then*

(i) *$(N, +)$ is abelian;*

(ii) *the affine plane A determined by N can be coordinatized by a field $(F, +, \cdot)$;*

(iii) *each $(\overline{B}_a, +, \cdot)$ is a field;*

(iv) *each $B_a = \{(x, mx) \mid x \in F\}$ for some $m \in F$, or $B_a = \{(0, x) \mid x \in F\}$.*

PROOF. Each $(\overline{B}_a, +, \cdot)$ is a near-field, hence $(N, +)$ is a p-group. Now $(\overline{B}_a, +)$ is contained in the center of $(N, +)$ for some $a \in N^*$, hence $(N, +)$ is abelian since $N = \overline{B}_a + \overline{B}_c$. A finite skew field is a field, and each $(\overline{B}_a, +, \cdot)$ is isomorphic to the coordinization skew field.

Examples. 1. Let $(F, +, \cdot)$ be a field. Define $+_\lambda$ $(\lambda \neq 0)$ by $a +_\lambda b = b$ if $a = 0$, $a +_\lambda b = a + (\lambda b)$ when $a \neq 0$. Then $(F, +_\lambda, \cdot)$ is a nontrivial integral planar system where $\equiv_m$ is discrete and $+_\lambda$ is not necessarily associative.

2. Let $(R\times R, +)$ be additive group of complex numbers. Define $\cdot$ by $(a, b)\cdot(c, d)=\|(a, b)\|(c, d)$ where $\|\cdot\|$ is any norm on $R\times R$. Then $(R\times R, +, \cdot)$ is an integral planar near-ring.

3. Let $(R\times R, +)$ be as in 2. Define $\cdot$ by $(a, b)\cdot(c, d)=(a, b)^{\wedge}(c, d)$ where $(a, b)^{\wedge}=0$ if $a=b=0$; otherwise $(a, b)^{\wedge}$ is the first nonzero coordinate. Then $(R\times R, +, \cdot)$ is an integral planar near-ring.

4. Let $(R\times R, +)$ be as in 2. Define $*$ by $(a, b)*(c, d)=(a, b)/|)a, b)|\cdot(c, d)$ where $|(a, b)|=(a^2+b^2)^{1/2}\neq 0$ and $\cdot$ denotes the usual multiplication of complex numbers. If $(a, b)=(0, 0)$, then $(a, b)*(c, d)=(0, 0)$. Then $(R\times R, +, \cdot)$ is an integral planar near-ring.

5. Table 1 defines a multiplication $\cdot$ on the cyclic group $(Z_5, +)$ such that $(Z_5, +, \cdot)$ is an integral planar near-ring. Note that $B_1=\{1, 4\}$, $B_2=\{2, 3\}$. Define $\overline{B}_i=B_i\cup\{0\}$ and $B_{ij}=\overline{B}_i+j$, $i=1, 2$; $j\in Z_5$. If

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	4	3	2	1
3	0	1	2	3	4
4	0	4	3	2	1

TABLE 1

we let $I=Z_5$, then the B_{ij} are circles of an inverse plane [3]. This example was obtained using a digital computer. (See [2].)

It is of interest to graph the left identities and the B_a in each of the Examples 2, 3, and 4.

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