AN ABELIAN p-GROUP WITHOUT THE ISOMORPHIC REFINEMENT PROPERTY[1]

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It is well known that a countable reduced abelian p-group G has the isomorphic refinement property, i.e. any two direct decompositions of G have isomorphic refinements, if and only if G has no elements of infinite height. In the uncountable case the situation is characteristically unclear. There do exist uncountable reduced p-groups with nonzero elements of infinite height having the isomorphic refinement property; any p-group whose first Ulm factor is torsion-complete and second Ulm factor is cyclic is such a group. And for uncountable p-groups with no elements of infinite height there exist sufficient conditions for the isomorphic refinement property, e.g., those of Crawley [3] and Warfield [4]. Yet the question has remained whether the isomorphic refinement property is possessed by all such groups. Here we answer this question in the negative by showing that *there exists an abelian p-group with no elements of infinite height having two direct decompositions that do not admit isomorphic refinements.*

The foregoing result is actually obtained as a corollary to the following theorem: *there exist three abelian p-groups K, L and M, each with no elements of infinite height, such that no Ulm invariant of K exceeds* 1, $K\oplus L\cong K\oplus M$, *yet* $L\not\cong M$. In particular, this shows that the cancellation theorem of Crawley [2] does not extend to the uncountable case.

To see how our first result follows from the second, let K, L and M be as above, and assume further that K has the isomorphic refinement property. We will show that the direct decompositions $K\oplus L\cong K\oplus M$ do not have isomorphic refinements. If they do, there exist groups K_i, L_i, K_i', M_i $(i=1, 2)$ such that

$$K = K_1 \oplus K_2 = K_1' \oplus K_2', \qquad L = L_1 \oplus L_2, \qquad M = M_1 \oplus M_2,$$

and

$$K_1 \cong K_1', \qquad K_2 \cong M_1, \qquad L_1 \cong K_2', \qquad L_2 \cong M_2.$$

Now by assumption, K has the isomorphic refinement property, and therefore there exist groups K_{ij}, K_{ij}' $(i, j=1, 2)$ such that

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$$K_i = K_{i1} \oplus K_{i2}, \qquad K'_i = K'_{i1} \oplus K'_{i2} \quad (i = 1, 2),$$

and

$$K_{ij} \cong K'_{ji} \qquad (i, j = 1, 2).$$

But all the Ulm invariants of K are at most 1, and $K_1 \cong K'_1$, so K_1 can have no nonzero Ulm invariant in common with either K_2 or K'_2. Therefore we must have

$$K_{12} = K'_{21} = K_{21} = K'_{12} = 0,$$

and consequently $K_2 = K_{22} \cong K'_{22} = K'_2$. This implies that $L_1 \cong M_1$, and we infer that $L = L_1 \oplus L_2 \cong M_1 \oplus M_2 = M$, a contradiction. Thus if K has the isomorphic refinement property, then $K \oplus L$ does not.

Throughout the proof of the second result the following notation is used. The cyclic subgroup generated by an element x in an abelian p-group G is denoted by $[x]$, and $G[p^n]$ and p^nG denote, respectively, the subgroups of those element of order at most p^n and those elements of height at least n. $E(G)$ denotes the ring of endomorphisms of the group G, and endomorphisms are written on the right. An endomorphism $\phi \in E(G)$ is called *small* if for each integer $e > 0$ there exists an integer $n > 0$ such that $(p^nG)[p^e]\phi = 0$. The small endomorphisms form an ideal of $E(G)$ which we denote by $E_s(G)$. Finally Z denotes the ring of integers.

Our proof requires the following theorem of Corner [1]. Let $\overline{B}$ be a torsion-complete abelian p-group with an unbounded countable basic subgroup B, and let Φ be a separable closed subring of $E(\overline{B})$ in the p-adic topology. Suppose further that Φ satisfies the condition

(C) if $\phi \in \Phi$ and $(p^n\overline{B})[p]\phi = 0$ for some integer $n > 0$, then $\phi \in p\Phi$. Then there exists a pure subgroup G of $\overline{B}$ which contains B, such that $E(G) = \Phi \oplus E_s(G)$.

For our purposes here, let A be a direct sum of cyclic groups, $A = \sum_{n<\infty} [a_n]$, where each a_n has order p^n. Set $B = A \oplus A$, and let $\overline{A}$ and $\overline{B}$ be, respectively, the torsion-completions of A and B. Let τ be that endomorphism of $\overline{A}$ defined by the rule

$$a_n\tau = pa_{n+1} \qquad (n = 1, 2, \cdots),$$

and let R be the subring of $E(\overline{A})$ generated by τ. Trivially $R \cong Z[\tau]$, τ transcendental. Identify the ring of all 2×2 matrices over R as a subring of $E(\overline{A} \oplus \overline{A}) = E(\overline{B})$, and let S be the subring of all matrices of the form

$$\begin{pmatrix} f_{11}(\tau) & (1-\tau^2)f_{12}(\tau) \\ f_{21}(\tau) & a + (1-\tau^2)f_{22}(\tau) \end{pmatrix} \qquad a \in Z, f_{ij}(\tau) \in R.$$

The ring S contains the matrices

$$\theta = \begin{pmatrix} \tau & 1-\tau^2 \\ 0 & 0 \end{pmatrix}, \qquad \phi = \begin{pmatrix} \tau & 0 \\ 1 & 0 \end{pmatrix},$$

$$\alpha = \theta\phi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \beta = \phi\theta = \begin{pmatrix} \tau^2 & \tau(1-\tau^2) \\ \tau & 1-\tau^2 \end{pmatrix};$$

and $\theta\phi\theta=\theta$, $\phi\theta\phi=\phi$. Consequently α and β are equivalent idempotents in S. Let $\overline{R}$ and $\overline{S}$ be, respectively, the p-adic closures of R in $E(\overline{A})$ and S in $E(\overline{B})$. It is easy to check that the ring $\overline{R}$ satisfies condition (C) with $\overline{B}$ replaced by $\overline{A}$, and from this and the purity of $\overline{S}$ in $E(\overline{B})$ it follows that $\overline{S}$ satisfies (C). Therefore by the theorem above there exists a pure subgroup G of $\overline{B}$ which contains B and such that $E(G)=\overline{S}\oplus E_s(G)$.

Since α and β are equivalent idempotents in $E(G)$,

$$G = G\alpha \oplus G(1-\alpha) = G\beta \oplus G(1-\beta)$$

and $G\alpha\cong G\beta$. Moreover, $B\alpha$ is a basic subgroup of $G\alpha$, and $A\cong B\alpha$, so that the nth Ulm invariant of $G\alpha$ is 1 for each $n=0, 1, 2, \cdots$. Suppose that $G(1-\alpha)\cong G(1-\beta)$. Then $1-\alpha$ and $1-\beta$ are equivalent idempotents in $E(G)$ and hence in $\overline{S}$, i.e. there exist endomorphisms $\mu, \nu \in \overline{S}$ with $\mu\nu = 1-\alpha$, $\nu\mu = 1-\beta$, $\mu\nu\mu = \mu$, $\nu\mu\nu = \nu$. Reduce all matrices in sight modulo p. Since $\alpha\mu=\nu\alpha=0$, we may set

$$\mu = \begin{pmatrix} 0 & 0 \\ f_1(\tau) & a+(1-\tau^2)f_2(\tau) \end{pmatrix}, \qquad \nu = \begin{pmatrix} 0 & g_1(\tau) \\ 0 & b+(1-\tau^2)g_2(\tau) \end{pmatrix}$$

where $a, b\in Z/pZ$ and $f_i(\tau)$, $g_i(\tau)$ are polynomials in τ over Z/pZ. Equate respectively the (1, 1) and (2, 1) entries in $\nu\mu=1-\beta$ to obtain

$$g_1(\tau)f_1(\tau) = 1-\tau^2, \qquad (b+(1-\tau^2)g_2(\tau))f_1(\tau) = -\tau.$$

The second equation requires that $g_2(\tau)=0$, so that $\tau\mid f_1(\tau)$, and this formula combined with the first equation gives $\tau\mid 1-\tau^2$, a contradiction. Therefore $G(1-\alpha)\not\cong G(1-\beta)$, and the proof is complete.

References

1. A. L. S. Corner, *On endomorphism rings of primary abelian groups*, (to appear).

2. P. Crawley, *The cancellation of torsion abelian groups in direct sums*, J. of Algebra 2 (1965), 432–442.

3. ———, *An isomorphic refinement theorem for certain abelian p-groups*, J. of Algebra 6 (1967), 376–387.

4. R. B. Warfield, Jr., *Complete abelian groups and direct sum decompositions*, Thesis, Harvard University, Cambridge, Mass., 1967.

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