PROOF OF A CONJECTURE OF HELSON[1]

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Let m_n denote the Haar measure of the torus T^n, the distinguished boundary of the unit polydisc U^n in the space of n complex variables. If f is holomorphic in U^n, define

$$f^*(z) = \lim_{r \to 1} f(rz) \tag{1}$$

for those $z \in T^n$ for which this radial limit exists. Here $z = (z_1, \cdots, z_n)$, $rz = (rz_1, \cdots, rz_n)$. The various H^p-norms in U^n, for $0 < p < \infty$, $n = 1, 2, 3, \cdots$, are defined by

$$\|f\|_{p,n} = \sup_{0<r<1} \left\{ \int_{T^n} |f(rz)|^p dm_n(z) \right\}^{1/p}. \tag{2}$$

As in one variable, the inequality

$$\log |f(0)| \leqq \int_{T^n} \log |f^*(z)| \, dm_n(z) \tag{3}$$

holds for every $f \in H^p(U^n)$. Define

$$\Delta(f) = \int_{T^n} \log |f^*(z)| \, dm_n(z) - \log |f(0)|. \tag{4}$$

For $f \in H^2(U^n)$, let $S(f)$ denote the H^2-closure of the set of all products Pf, where P ranges over the polynomials in n variables; $S(f)$ is the *invariant subspace of* $H^2(U^n)$ *generated by* f.

A very well-known theorem of Beurling states (in one variable) that

$$S(f) = H^2(U) \quad \textit{if and only if} \quad \Delta(f) = 0. \tag{5}$$

One of these implications holds equally well for several variables, as has been known for quite some time to Helson and Lowdenslager: *If* $f \in H^2(U^n)$ *and* $S(f) = H^2(U^n)$, *then* $\Delta(f) = 0$. Here is a sketch of a simple proof: (i) $\Delta(Pf) = \Delta(P) + \Delta(f) \geqq \Delta(f)$ for all P. (ii) Δ is an upper semicontinuous function on $H^2(U^n)$. (iii) Therefore $\Delta(g) \geqq \Delta(f)$ for every $g \in S(f)$.

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Helson has conjectured [1, p. 23] that the converse is false for $n=2$ (hence also for $n>2$). (Actually, Helson stated the problem somewhat differently, in terms that involve only the boundary values of the functions under consideration.) This conjecture is correct:

THEOREM. *There exists a function* $f\in H^2(U^2)$ *such that* $\Delta(f)=0$ *but* $S(f)\neq H^2(U^2)$.

The proof depends on the following two observations.

(I) *If* $F\in H^\infty(U)$, *if* F *has no zero in* U, *and if* $f\in H^\infty(U^2)$ *is defined by*

$$f(z_1, z_2) = F((z_1 + z_2)/2), \tag{6}$$

then $\Delta(f)=0$.

(II) *Associate to each* $f\in H^2(U^2)$ *the function*

$$(\Psi f)(\lambda) = f((1+\lambda)/2\ (1+\lambda)/2) \qquad (\lambda \in U). \tag{7}$$

If $0<p<\frac{1}{2}$, *there is a constant* $C_p<\infty$ *such that*

$$\|\Psi f\|_{p,1} \leqq C_p\|f\|_{2,2}. \tag{8}$$

Thus Ψ maps $H^2(U^2)$ into $H^p(U)$ if $p<\frac{1}{2}$. Note that Ψf is essentially the restriction of f to a certain disc in U^2 which touches T^2 at just one point.

PROOF OF (I). If $|\alpha|=1$, $z\to\alpha z$ is a measure-preserving map of T^2 onto T^2. Hence

$$\int_{T^2} dm_2(z)\int_T \log|f^*(\alpha z)|\,dm_1(\alpha) = \int_{T^2}\log|f^*(z)|\,dm_2(z), \tag{9}$$

as is seen by interchanging the integrations on the left. If $z=(z_1,z_2)\in T^2$, if $z_1\neq z_2$, and if $2w=z_1+z_2$, then $|w|<1$, so that

$$\log|F(0)| = \int_T \log|F(\alpha w)|\,dm_1(\alpha).$$

This says that the inner integral on the left of (9) is equal to $\log|f(0)|$ whenever $z_1\neq z_2$, which is true for almost all $z\in T^2$. Hence $\Delta(f)=0$.

PROOF OF (II). For simplicity, assume $\|f\|_{2,2}=1$. Apply the Schwarz inequality to the Cauchy formula

$$f(\zeta,\zeta) = \int_{T^2}\frac{f^*(z_1, z_2)}{(1-\bar z_1\zeta)(1-\bar z_2\zeta)}\,dm_2(z)$$

to obtain the estimate

$$|f(\zeta,\zeta)| \leqq \left\{\int_{T^2} |1-\bar{z}_1\zeta|^{-2}|1-\bar{z}_2\zeta|^{-2}dm_2(z)\right\}^{1/2}$$

$$= \int_T |1-\bar{w}\zeta|^{-2}dm_1(w) = (1-|\zeta|^2)^{-1}$$

if $|\zeta|<1$. For $\lambda = re^{i\theta}$, $0<r<1$, it follows that

$$|(\Psi f)(\lambda)| \leqq \{1-|(1+\lambda)/2|^2\}^{-1} \leqq \{r\sin^2(\theta/2)\}^{-1}$$

which gives (8) with

$$C_p = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin(\theta/2)|^{-2p}d\theta\right\}^{1/p}.$$

PROOF OF THE THEOREM. Put $F(\lambda) = \exp\{(\lambda+1)/(\lambda-1)\}$ and associate f with F as in (I). Then $\Delta(f)=0$.

Fix p, $0<p<\frac{1}{2}$. If P is any polynomial in two variables, (II) gives

$$\|1-Pf\|_{2,2} \geqq C_p^{-1}\|1-\Psi P\cdot\Psi f\|_{p,1}. \tag{10}$$

Note that $(\Psi f)(\lambda) = e^{-1}F^2(\lambda)$. Thus $e\Psi f$ is a nontrivial inner function in U. Since multiplication by an inner function is an isometry in $H^p(U)$ (relative to the metric given by $\|g-h\|_{p,1}^p$ if $p<1$) one sees that $H^p(U)\Psi f$ is a closed subspace of $H^p(U)$ which does not contain 1. The right side of (10) is therefore bounded below by some positive constant, and so (10) implies that 1 is not in $S(f)$. Hence $S(f)\neq H^2(U^2)$.

REFERENCE

1. Henry Helson, *Lectures on invariant subspaces*, Academic Press, New York, 1964.

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