GENERIC ONE-PARAMETER FAMILIES OF VECTOR FIELDS ON TWO-DIMENSIONAL MANIFOLDS

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Introduction. This paper is an announcement of a study on the theory of the topological variation of the phase space of one-parameter families of vector fields (ordinary differential equations, flows). This theory, sometimes called bifurcation theory, has been developed since H. Poincaré from several points of view; see, for example, [1], [2], [3], [4], [5]. Here, we will be mainly interested in a collection of one-parameter families of vector fields which has the following properties: (a) it is large with respect to all the families, and (b) its elements exhibit a topological variation which is amenable to simple description.

Collections with properties (a) and (b) are currently called "generic" and were introduced in the qualitative global analysis of differential equations by M. Peixoto [6], S. Smale [8], and I. Kupka [11]. See S. Smale [9] for a thorough monography on this field.

The geometry of the set Σ of structurally stable vector fields and the study of "generic" one-parameter families of vector fields are closely related. A vector field is structurally stable if its phase space does not change topologically under small perturbations; a one-parameter family of vector fields exhibits: the simpler a phase-space topological variation, the larger the intersection it has with Σ, or equivalently, the smaller the intersection it has with its complement —the set of nonstructurally stable vector fields.

The importance of the set of nonstructurally stable vector fields for the study of the topological variation of the phase space of vector fields was noticed by A. Andronov and E. Leontovich. In [12] they defined the concept of first-order structural stability as a possible guide to pursue such study.

In compact two-dimensional manifolds, the only case considered here, the set of first-order structurally stable vector fields is an imbedded Banach manifold of class C^1 and codimension one of the Banach manifold of vector fields; see [13]. This fact, although (as follows from [14]) not sufficient to describe completely the "generic"

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one-parameter families of vector fields, motivates the main result of this paper: There is an immersed Banach submanifold, Σ_1, of class C^1 and codimension one of the Banach manifold of vector fields with C^r-topology, $r\geqq 4$, such that the "generic" one-parameter families of vector fields intersect it transversally at points where they are not vector fields of Kupka-Smale type. This result is stated in a precise and complete form in Theorems 1 and 2; it answers questions raised by M. Peixoto.

For the proof of these theorems, which will be published elsewhere, the characterization of Σ given by M. Peixoto [7] and the approximation techniques of I. Kupka and S. Smale, [11], [10], are essential.

Preliminary definitions. Let M^2 be a C^∞ two-dimensional compact manifold. Call $\mathfrak{X}^r$ the space of C^r-tangent vector fields defined on M^2 under the C^r-topology. Denote by ϕ_X: $R\times M^2\rightarrow M^2$ the C^r-flow induced by $X\in\mathfrak{X}^r$.

DEFINITION 1. Let X and $Y\in\mathfrak{X}^r$, $r\geqq 1$; they are said to be *topologically equivalent* if there is a homeomorphism of M^2 onto itself sending trajectories of X onto trajectories of Y. If X has a neighborhood $N(X)$ in $\mathfrak{X}^r$ such that it is topologically equivalent to every $y\in N(X)$, X is said to be *structurally stable.*

Denote by Σ^r the set of structurally stable vector fields and denote by $\mathfrak{X}_1^r$ its complement relative to $\mathfrak{X}^r$ endowed with the induced C^r-topology.

DEFINITION 2. A continuous function ξ: $J=[a, b]\rightarrow\mathfrak{X}^r$ is called a *one-parameter family of vector fields.* A point $\lambda_0\in J$ is said to be an *ordinary value* of ξ if for any $\epsilon>0$, it has a neighborhood $N(\lambda_0)$ such that $\xi(\lambda)$ is topologically equivalent to $\xi(\lambda_0)$ for every $\lambda\in N(\lambda_0)$; λ_0 is called a *bifurcation value* of ξ if it is not an ordinary value of ξ.

Quasi-generic vector fields. Let $p\in M^2$ be a singular point of $X\in\mathfrak{X}^r$, i.e. $X(p)=0$. For $r\geqq 1$ and any $V\in\mathfrak{X}^r$, $[V, X](p)$ depends only on $V(p)$. This remark makes it possible to define an endomorphism L_p of the tangent space T_p at p. L_p is defined as follows: if $v\in T_p$, let $V\in\mathfrak{X}^r$ be any extension of it and define $L_p(v)=[V, X](p)$. The determinant and trace of L_p are denoted respectively by $\Delta(X, p)$ and $\sigma(X, p)$. If L_p is an isomorphism, the singular point p of X is said to be *simple*. It is said to be *generic* if the eigenvalues of L_p have nonvanishing real parts; if the eigenvalues of L_p are real and have opposite sign, p is called a *saddle*; if they have equal sign, p is called a *node*; if the eigenvalues have nonvanishing imaginary parts, p is called a *focus*.

Let λ_1 and λ_2 be the eigenvalues of L_p, assume that they are real and different and denote respectively by T_1 and T_2 the eigenspaces associated with them. Call respectively π_1 and π_2 the projections of T_p onto T_1 and T_2 associated with the splitting $T_p = T_1 \oplus T_2$.

DEFINITION 3. A singular point p of $X \in \mathfrak{X}^r$, $r \geqq 2$, is said to be a *saddle-node* if only one of the eigenvalues of L_p, say λ_1, vanishes and $\Delta_1(X, p, v) \neq 0$, $v \in T_1$, $v \neq 0$, $\Delta_1(X, p, v)v = \pi_1[V, [V, X]](p)$ where V is any C^r-extension of v. It can be proved that the saddle-node is well defined.

Suppose that the eigenvalues of L_p have nonvanishing imaginary parts. Let I be a C^∞ arc on M^2 with one extreme at p. It is classical that the flow ϕ_X defines homeomorphism ρ_X: $I_0 \to I$ on an I-neighborhood I_0 of p; ρ_X assigns to $q \in I_0$ the point where the semiorbit $\phi_X(t, q)$, $t > 0$, meets I for the first time; ρ_X is of class C^r if X is so.

DEFINITION 4. (a) A singular point p of $X \in \mathfrak{X}^r$, $r \geqq 3$, is said to be a *composed focus* if $\rho_X^1(p) = 1$ and $\rho_X^3(p) \neq 0$.

(b) A singular point p of $X \in \mathfrak{X}^r$, $r \geqq 3$, is said to be *quasi-generic* if it is either a saddle-node (Definition 3) or a composed focus.

Suppose that X has a nontrivial periodic orbit through p and let I be a C^∞ arc transversal to X which has p as interior point. The flow ϕ_X defines a homeomorphism π_X: $I_0 \to I$ defined on an I-neighborhood I_0 of p; $\pi_X(q)$ is the point where the semiorbit $\phi_X(t, q)$, $t > 0$, meets I for the first time. π_X is of the same class of differentiability as X and is currently called the Poincaré transformation associated with X, p, and I.

DEFINITION 5. A periodic orbit of $X \in \mathfrak{X}^r$, $r \geqq 1$, passing through p is called *generic* if $|\pi_X^1(p)| \neq 1$; it is called *quasi-generic* if either $\pi_X^1(p) = 1$, $r \geqq 2$, and $\pi_X^{(2)}(p) \neq 0$ or $\pi_X^1(p) = -1$, $r \geqq 3$, and $(\pi_X \circ \pi_X)^{(3)}(p) \neq 0$.

Denote by $\omega_X(p)$ (resp. by $\alpha_X(p)$) the ω-limit (resp. the α-limit) set of the orbit of X which passes through p.

DEFINITION 6. (a) If in every neighborhood of p there are points q such that $\omega_X(q) \neq \omega_X(p)$ (resp. $\alpha_X(q) \neq \alpha_X(p)$), then the orbit of X passing through p is called ω separatrix (resp. α separatrix).

(b) An α and (or) ω separatrix is said to be a *saddle connection* (saddle separatrix) if its α and (or) ω-limit sets are saddle or saddle-node singular points.

(c) If the α and ω-limit sets of a saddle connection of X are equal to a saddle point q, such that $\sigma(X, q) \neq 0$, then the set formed by the saddle connection and q is called a *simple graph*.

DEFINITION 7. Assume $r \geqq 3$. (a) Q_1^r is defined to be the set of C^r-vector fields which have one quasi-generic singular point, have only generic periodic orbits, and do not have saddle connections.

(b) Q_2^r is defined to be the set of C^r-vector fields which have one quasi-generic periodic orbit as unique nongeneric periodic orbit, have only generic singular points, and do not have saddle connections.

(c) Q_3^r is defined to be the set of C^r-vector fields which have one saddle connection which, in case of being a self-connection is part of a simple graph, have only generic singular points and generic periodic orbits.

(d) $[\text{K-S}]^r$ (Kupka-Smale vector fields) is defined to be the set of C^r-vector fields which have only generic singular points, only generic periodic orbits, and do not have saddle connections.

(e) P^r is defined to be the set of C^r-vector fields X for which $\omega_X(p)$ and $\alpha_X(p)$, for every $p \in M^2$, can only be singular points, periodic orbits, or graphs, i.e. vector fields for which Poincaré-Bendixon theorem is valid.

Banach submanifolds. Let B be a Banach manifold of class C^∞ in the sense of S. Lang [**15**, p. 16]; obviously, $\mathfrak{X}^r$ belongs to this category.

DEFINITION 8. (a) A subset $S \subset B$ is said to be an *imbedded Banach manifold of class* C^s *and codimension* k *of* B if every $p \in S$ has a neighborhood $N(p)$ such that there exists a C^s-function $f\colon N(p) \to R^k$ such that df_p has maximum rank and $f^{-1}(0) = S \cap N(p)$.

(b) $S \subset B$ is said to be an *immersed Banach manifold of class* C^s *and codimension* k *of* B if there is a countable family $\{S_i\}_{i\in N}$ of imbedded Banach manifolds of class C^s and codimension k of B such that $S_i \subset S_{i+1}$, $i \in N$, and $S = \bigcup_{i=1}^\infty S_i$.

It is easily verified that Definition 7 implies the definitions of the same concepts, as given in [**13**, p. 19 and 20].

DEFINITION 9. Φ^r is defined to be the set of one parameter families of vector fields $\xi\colon J \to \mathfrak{X}^r$ of class C^1, under the C^1-topology. Obviously, Φ^r is a Banachable space.

The results. We assume $r \geqq 4$.

THEOREM 1. (a) $\Sigma_1^r = P^r \cap (\bigcup_{i=1}^3 Q_i^r)$ *is an immersed Banach manifold of class* C^1 *and codimension one of* $\mathfrak{X}^r$.

(b) Σ_1^r *is dense in* $\mathfrak{X}_1^r$.

(c) *If* $X \in \Sigma_1^r$, *there is a* Σ_1^r-*neighborhood* $N(X)$ *of* X *such that* X *is topologically equivalent to every* $Y \in N(X)$.

THEOREM 2. *Let* Γ^r *be the set of one-parameter families of vector fields* $\xi \in \Phi^r$ *such that*

(a) $\xi(J) \subset \Sigma_1^r \cup [K-S]^r$.

(b) ξ *is transversal to* Σ_1^r.

(c) *The set of bifurcation values of* ξ *is a closed nowhere dense set of* J; *it coincides with* $J - \xi^{-1}(\Sigma^r)$.

Then, Γ^r contains a residual subset of Φ^r; in particular, it is a dense subset of Φ^r.

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