ANALYTIC DOMINATION BY FRACTIONAL POWERS OF A POSITIVE OPERATOR

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Introduction. Let A be an (unbounded) linear operator on a Banach space $\mathfrak{H}$. An *analytic vector* for A is an element $u\in\mathfrak{H}$ such that $A^n u$ is defined for all n and

$$\sum_{n=0}^{\infty} \frac{\|A^n u\|}{n!} t^n < \infty$$

for some $t>0$, i.e. the power series expansion of $e^{tA}u$ is defined and has a positive radius of absolute convergence.

Nelson [2] introduced and studied the notion of *analytic domination* of one operator (or a family of operators) by another: A analytically dominates the operator X if every analytic vector for A is an analytic vector for X. In §1 we announce an analytic domination theorem; the hypotheses were suggested by Nelson's treatment of Lie algebras of skew-symmetric operators in [2], while the conclusion was suggested by some results of Kotake and Narasimhan [1]. We apply our theorem in §2 to the characterization of analytic vectors for a unitary representation of a Lie group.

1. **Analytic domination.** Let $\mathfrak{H}$ be a complex Hilbert space, and A a positive, selfadjoint operator on $\mathfrak{H}$, which we normalize so that $A\geqq I$. If α is a complex number, the operator A^α is defined via the operational calculus for selfadjoint operators, and $\mathfrak{D}(A^\alpha)\subseteq\mathfrak{D}(A^\beta)$ if $\operatorname{Re}\alpha\geqq\operatorname{Re}\beta$. (For any operator T on $\mathfrak{H}$, $\mathfrak{D}(T)$ will denote its domain of definition.) Let

$$\mathfrak{H}^\infty = \bigcap_{n=1}^{\infty} \mathfrak{D}(A^n)$$

(the C^∞-vectors for A). Then we have the following analytic domination criterion: $(\operatorname{ad}X(A)=XA-AX)$.

THEOREM 1. *Let $X\colon \mathfrak{H}^\infty\to\mathfrak{H}^\infty$ be symmetric or skew-symmetric. Suppose that for some α, $0<\alpha<1$,*

$$\|Xu\|\leqq\|A^\alpha u\|, \tag{1}$$

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(2) $$\|(\mathrm{ad}X)^n(A)u\| \leqq n!\,\|Au\|$$

for all $u \in \mathfrak{H}^\infty$. *Then every analytic vector for* A^α *is an analytic vector for* X.

The proof of Theorem 1 shows slightly more, namely

COROLLARY 1.1. *Suppose* $u \in \mathfrak{H}^\infty$ *and* $\|A^n u\| \leqq M^n n^{n/\alpha}$, *for some constant* M. *Then* u *is an analytic vector for* X, *and there exists a constant* C *depending only on* M *and* α *such that* $\|X^n u\| \leqq C^n n!$.

If we eliminate the assumption of symmetry or skew-symmetry on X, then the proof of Theorem 1 yields (we use the notation $(u|v)$ for the inner product in $\mathfrak{H}$):

COROLLARY 1.2. *Suppose* X: $\mathfrak{H}^\infty \to \mathfrak{H}^\infty$ *and* X *has an adjoint*

$$X^+ \colon \mathfrak{H}^\infty \to \mathfrak{H}^\infty$$

(*i.e.* $(Xu|v) = (u|X^+v)$ *for* $u, v \in \mathfrak{H}^\infty$). *Suppose conditions* (1) *and* (2) *of Theorem* 1 *are satisfied by both* X *and* X^+. *Then the conclusions of Theorem* 1 *and Corollary* 1.1 *hold for* X (*and for* X^+).

REMARKS. The case $\alpha = 0$ of the theorem is trivial, since it implies X bounded. The case $\alpha = 1$ is Nelson's analytic domination theorem, [2, Corollary 3.2]. Our proof, roughly speaking, proceeds by first showing that one may replace A by A^α in (2), and then applying Nelson's theorem relative to A^α and X.

The idea of the proof is quite simple: we observe that A^α can be expressed in terms of an integral involving $A(A+\lambda)^{-1}$, $\lambda \geqq 0$; hence we can estimate $(\mathrm{ad}X)^n(A^\alpha)$ in terms of $(\mathrm{ad}X)^n[A(A+\lambda)^{-1}]$. The precise inequalities, however, are somewhat subtle. Direct norm estimates lead to a logarithmically divergent integral; we must use the symmetry of X and A together with interpolation on suitable fractional quadratic norms in order to obtain the needed *a priori* estimates for Nelson's theorem.

2. **Analytic vectors for unitary representations.** Let G be a Lie group, $\mathfrak{G}$ its Lie algebra, and suppose U is a continuous unitary representation of G on a Hilbert space $\mathfrak{H}$. To every vector $v \in \mathfrak{H}$ we associate its trajectory $\tilde{v}$ under U. We say that v is a C^∞ (resp. analytic) vector if $\tilde{v}$ is infinitely differentiable (resp. real analytic) as an $\mathfrak{H}$-valued function on G, and we denote the corresponding subspaces of $\mathfrak{H}$ by $\mathfrak{H}^\infty$ and $\mathfrak{H}^\omega$. On $\mathfrak{H}^\infty$, U defines a representation of $\mathfrak{G}$ by skew-symmetric operators. (See [2].)

Let $X_1, \cdots, X_d$ be a basis for $\mathfrak{G}$, and set $\Delta = \sum_{k=1}^{d} X_k^2$. The operator $U(1-\Delta)$ is symmetric on $\mathfrak{H}^\infty$ and its closure, which we denote by A, is a positive selfadjoint operator, $A \geqq 1$ [2]. Furthermore the space $\mathfrak{H}^\infty$ of infinitely differentiable vectors for the representation is definable in terms of A, namely

$$\mathfrak{H}^\infty = \bigcap_{n=1}^{\infty} \mathfrak{D}(A^n)$$

([2, Corollary 9.3]). Nelson also proved that every analytic vector for A was in $\mathfrak{H}^\omega$, by employing his analytic domination theorem. By using our Theorem 1, we can obtain a sharper result. Set $B = A^{1/2}$. Then we have

THEOREM 2. *$\mathfrak{H}^\omega$ is precisely the set of analytic vectors for B.*

Using the more explicit estimates of Corollary 1.1, we obtain

COROLLARY 2.1 *Let $v \in \mathfrak{H}^\infty$. Then $v \in \mathfrak{H}^\omega$ if and only if there exists a constant M such that*

$$\| U(\Delta)^n v \| \leqq M^n (2n)!$$

for all n. In this case there exists a neighborhood V of 0 in $\mathfrak{G}$ depending only on M such that

$$\sum_{n=0}^{\infty} \frac{1}{n!} U(X)^n v$$

is absolutely convergent for $X \in V$.

REFERENCES

1. T. Kotake and M. S. Narasimhan, *Regularity theorems for fractional powers of a linear operator*, Bull. Soc. Math. France **90** (1962), 449–471.

2. E. Nelson, *Analytic vectors*, Ann. of Math. **70** (1959), 572–615.

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