ON MEAN-PERIODICITY

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Introduction. Wiener's classical results on the closure of the translates of a function belonging to a Lebesgue class lead to the notion of mean-periodicity: A function f belonging to a given space M of functions defined on a Euclidean space is said to be *mean-periodic* in the space M if the closed linear span of the translates of f falls short of the entire space M. This notion clearly depends very strongly upon the function space M under consideration. The space M most thoroughly studied to date is the space of all continuous functions on the real line R under the topology of uniform convergence on compact sets [4], [5], [8].

It is the purpose of the present note to outline some new results concerning spectral analysis and synthesis in translation invariant subspaces connected with a certain space of functions holomorphic on R.

A certain function space and its dual. We consider at first the following space M. Let $f(x+iy)$ denote a function holomorphic in a closed strip $|y| \leqq 1/h$ for some $h < \infty$ such that

$$\sup_{|y| \leqq 1/h} \left\| e^{|x|/k} f(x+iy) \right\| < \infty \tag{1}$$

for every $k > 0$, where the norm appearing is that of $L^2(R)$. For h and k fixed, let S_k^h denote the Banach space formed by all such functions f with norm given by the expression (1). The space M is then the set

$$\bigcup_{h<\infty} \bigcap_{k>0} S_k^h$$

with the natural topology

$$M = \operatorname*{ind\,lim}_{h \to \infty} \operatorname*{proj\,lim}_{k \to 0^+} S_k^h.$$

The space M resembles spaces considered by Gelfand and Silov [2], [3] and by Roumieu [7], but it does not coincide with any of these. The space M does occur amongst the spaces investigated by Palamodov [6].

It turns out that M possesses several convenient properties as a topological vector space. M is a Montel space and hence reflexive. M

is bornological. M is a regular inductive limit; that is, every bounded set in M is contained in a space

$$S^n = \operatorname{proj}\lim_{k\to 0^+} S_k^n$$

for some $n>0$ and bounded in S^n. This latter fact allows us to make effective use of the strong topology in the dual space M', defined by uniform convergence on the bounded sets of M.

The Fourier transformation $\mathfrak{F}$ effects an isomorphic mapping of the space M on

$$\mathfrak{F}M = \operatorname{ind}\lim_{k\to\infty}\ \operatorname{proj}\lim_{h\to 0^+} S_k^h$$

(Cf. [6, p. 328].)

The elements λ belonging to the dual space M' have the following structure

$$\lambda = \sum_{p,q} D^p[x^q g_{pq}(x)], \qquad g_{pq} \in L^2(R), \tag{2}$$

where the sum over nonnegative integers p and q is in general infinite and where

$$H^p p! K^q q! \|g_{pq}\| = O(1) \qquad (p, q \to \infty)$$

for all $H<\infty$ and some $K>0$. The series (2) converges strongly in M' and the action of λ on a function of M is according to the formula

$$\langle \lambda, f\rangle = \sum_{p,q} (-1)^p \int_R x^q g_{pq}(x) D^p f(x)dx, \qquad f \in M.$$

The function

$$z \to \phi_\lambda(z) = \sum_{p,q} (iz)^p (iD_z)^q \int_R \frac{G_{pq}(t)}{t-z}\,dt, \qquad G_{pq} = \mathfrak{F}g_{pq},$$

(the indicator of Fantappié of $\mathfrak{F}\lambda$) is of importance because the Fourier transform[1] $\mathfrak{F}\lambda$, $\lambda\in M'$, coincides with the linear form

$$F \to \int \phi_\lambda(s)F(s)ds, \qquad F = \mathfrak{F}f,\ f \in M, \tag{3}$$

where the integral is extended over two parallel lines in the upper and lower half-planes, situated outside of the strip $|y| \leqq 1/K$; $K=K_\lambda$ is

[1] Recall that the Fourier transform $\mathfrak{F}\lambda$ is defined through $\langle\mathfrak{F}\lambda, \mathfrak{F}f\rangle = 2\pi\langle\lambda, f\rangle$, where $\mathfrak{F}f(s) = \int_R e^{isx} f(x)\,dx$, $(f\in M)$.

the constant associated with the distribution λ. We have here a means of handling effectively complex spectra contained in a horizontal strip about the real axis.

It is not permissible to identify $\mathfrak{F}\lambda$ with ϕ_λ because several different functions ϕ_λ can define the same form (3) (cf. [7], [1]). However, any two such functions ϕ_λ must differ by an entire function of a certain class.

Mean-periodic distributions of M'. We designate as invariant subspaces those closed subspaces of the dual space M' which are invariant under all translations by $\xi \in R$.

PROPOSITION 1. *If $V' \neq (0)$ is an invariant subspace of M', then V' necessarily contains at least one character, $x \to e^{isx}$, s complex.*

The orthogonal space $(V')^\perp$ is an invariant subspace of M, and the spectrum of V' consists of all common zeros (with multiplicity) of the Fourier transforms $\mathfrak{F}f$ when f runs over $(V')^\perp$.

PROPOSITION 2. *The system of exponential-monomials contained in an invariant subspace V' has the property that no exponential-monomial of the system belongs to the closed linear span of the others.*

This fact allows us to write down a formal series for each mean-periodic $\lambda \in M'$:

$$\lambda \sim \sum_k \sum_{j=0}^{p_k-1} C_{jk}(ix)^j e^{is_k x}, \tag{4}$$

where s_k denotes the points of the spectrum of λ, p_k the multiplicity of s_k, and where the coefficients C_{jk} are given through the following rule (L. Schwartz [8]):

$$C_{jk} = \langle \lambda, n_{jk} \rangle, \qquad \mathfrak{F}n_{jk} = N_{jk},$$
$$N_{jk}(s) = N(s)\{\text{polar part of } (s - s_k)^j/j!N(s)\}, \quad j = 0, 1, \cdots, p_k - 1,$$
$$N = \mathfrak{F}n, \qquad \langle \lambda, n \rangle = 0, \qquad n \in M.$$

A major result is that the C_{jk} do not depend upon the choice of $n \in M$ orthogonal to λ, but are intrinsically connected with λ.

PROPOSITION 3. *The coefficients C_{jk} are uniquely determined by the distribution λ as follows:*

$$C_{jk} = \textit{residue of } \phi_\lambda(s)\ ((s - s_k)^j/j!) \textit{ at } s = s_k.$$

and $C_{jk} = 0$ in all other cases.

PROPOSITION 4. *Every mean-periodic distribution* λ *belonging to the space* M' *is a strong limit of a generalised sequence of linear combinations of exponential monomials associated with the spectrum of* λ.

The formal series (4) can be interpreted as an interpolation series and the remainder calculated. This calculation, taken with the above double characterisation of the coefficients C_{jk}, yields the more precise result:

THEOROM 5. λ *is a mean-periodic distribution belonging* to M′, *then the formal series* (4) *converges to* λ *in the strong topology of* M′.

Mean-periodic functions in the space $\mathfrak{F}M$. The cospectrum of a function belonging to the space $\mathfrak{F}M$ is the set of real zeros of its Fourier transform. The problem of spectral synthesis is to decide whether every function whose cospectrum contains the cospectrum of a given function f is a limit of translations of f in $\mathfrak{F}M$. It is familiar that such questions reduce to questions of divisibility in the Fourier transform space, which is M in the present case. Here it is possible to divide out the real zeros of a function of M by a Blaschke product in a horizontal strip. The necessary minorations in order to have a quotient in M result by considering the harmonic measure in a slit strip.

PROPOSITION 6. *Let f belong to* $\mathfrak{F}M$. *Every function of* $\mathfrak{F}M$ *whose cospectrum contains the cospectrum of f is a limit of a generalised sequence of linear combinations of translates of f.*

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