ON THE CHARACTERISTIC HOMOMORPHISM OF A DISCRETE UNIFORM SUBGROUP OF A NILPOTENT LIE GROUP

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1. In [2] A. Grothendieck quotes an example, due to J. P. Serre, of a complex algebraic nilpotent Lie group G and a discrete uniform subgroup $\alpha: \Gamma \subset G$, such that the characteristic homomorphism induced in the rational cohomology of the classifying spaces

$$B\alpha^*: H^*(B_G, \mathbf{Q}) \to H^*(B_\Gamma, \mathbf{Q}) \cong H^*(\Gamma, \mathbf{Q})$$

is not trivial. The purpose of this note is to generalize this example to nilpotent Lie groups admitting a discrete uniform subgroup.

Let G' be a simply connected and connected nilpotent (real) Lie group of dimension n with center $Z(G')$ and $\alpha': \Gamma' \subset G'$ a discrete uniform subgroup of G' (i.e., G'/Γ' compact). Then $A = \Gamma' \cap Z(G')$ is a free abelian group of rank $m = \dim Z(G') > 0$. Consider $G = G'/A$ and $\Gamma = \Gamma'/A$ with the natural inclusion $\alpha: \Gamma \subset G$ and the induced map in classifying spaces $B\alpha: B_\Gamma \to B_G$. It is clear that Γ is a discrete uniform subgroup of G.

THEOREM 1.1. *The rational characteristic homomorphism*

$$B\alpha^*: H^*(B_G, \mathbf{Q}) \to H^*(B_\Gamma, \mathbf{Q}) = H^*(\Gamma, \mathbf{Q})$$

is trivial (i.e. zero in positive dimensions) if and only if G' is the vector group $\mathbf{R}^n$.

Since G' is contractible, $G = G'/A$ is a space of type $K(A, 1)$ and hence the cohomology of its classifying space is given by $H^*(B_G, \mathbf{Z}) = \mathbf{Z}[x_1, \cdots, x_m]$, $\deg(x_k) = 2$. Moreover $B\alpha$ is the classifying map of the *flat* G-bundle $\eta: E_\Gamma \times_\Gamma G \to B_\Gamma$, where E_Γ is the total space of the universal Γ-bundle [4], [5]. The classes $x_k(\eta) = B\alpha^*(x_k) \in H^2(\Gamma, \mathbf{Z})$ are then the characteristic classes over $\mathbf{Z}$ of the G-bundle η.

COROLLARY 1.2. *The integral characteristic classes $x_k(\eta)$, $k = 1, \cdots, m$ of the flat G-bundle η are torsion classes if and only if G' is the vector group $\mathbf{R}^n$.*

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Since B_Γ can be realized as a compact manifold (§4), it has the homotopy type of a finite CW-complex; hence 1.2 is just a reformulation of 1.1.

Theorem 1.1 is in sharp contrast to [**4**, Theorems 2.2, 3.4, 3.5] and [**2**, Theorem 7.1] where sufficient conditions for the triviality of the rational characteristic homomorphism were given and it increases the number of examples [**8**], [**2**, 7.5] of flat principal bundles with non-trivial rational characteristic classes.

The proof of Theorem 1.1 will be given in §§2 and 3. In §4 we will consider a particular realization of B_Γ as a compact manifold and examine the question whether the tangent bundle of this manifold is associated to the bundle η in Corollary 1.2.

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2. **Proof of Theorem** 1.1. The "if" part of the theorem is trivial. In fact, $G' = \boldsymbol{R}^n$ implies $\Gamma = \{1\}$. The "only if" part will be proved in the formulation of Corollary 1.2. The characteristic classes $x_k(\eta) = B\alpha^*(x_k) \in H^2(\Gamma, \boldsymbol{Z})$ can be given various interpretations which we list in the following

LEMMA 2.1 [**5**, PROPOSITION 4.17], [**10**, p. 189]. *The following elements of* $H^2(\Gamma, A)$, $A \cong \pi_1(G) \cong \boldsymbol{Z}^m$ *are equal up to sign:*

(i) $x(\eta) = (x_k(\eta))_{k=1,\cdots,m}$.

(ii) $\tau_\eta(\iota)$, *where* ι *is the fundamental class in* $H^1(G, \pi_1(G)) \cong \mathrm{Hom}(\pi_1(G), \pi_1(G))$ *corresponding to the identity homomorphism and* τ_η *is the transgression in the bundle* η.

(iii) $\mathfrak{o}(\eta)$, *the primary obstruction to a cross section in* η.

(iv) $\delta(\alpha)$, *where* δ *is the coboundary in the non-abelian cohomology sequence*

$$* \to H^1(\Gamma, A) \to H^1(\Gamma, G') \to H^1(\Gamma, G) \xrightarrow{\delta} H^2(\Gamma, A)$$

associated to the universal covering sequence

$$0 \to A \to G' \to G \to 1$$

of G.

(v) $\phi(E)$, *the characteristic cohomology class of the central extension*

$$E: 0 \to A \underset{i}{\to} \Gamma' \underset{q}{\to} \Gamma \to 1.$$

REMARK 2.2. Lemma 2.1 only makes sense for specific isomorphisms $H^*(B_G, \boldsymbol{Z}) \cong \boldsymbol{Z}[x_1, \cdots, x_m]$ and $\pi_1(G) \cong \boldsymbol{Z}^m$. These isomorphisms will be exhibited at the end of §3.

By Lemma 2.1 it is sufficient to show that G' is the vector group $\boldsymbol{R}^n$ if the class $\phi(E)$ is a torsion class. The spectral sequence of the extension E determines a five-term exact sequence [**3**, Theorem 4.2]

$$(2.3)\qquad \begin{aligned} &0 \to H^1(\Gamma, A) \to H^1(\Gamma', A) \\ &\underset{i^*}{\to} H^1(A, A) \underset{\tau}{\to} H^2(\Gamma, A) \underset{q^*}{\to} H^2(\Gamma', A) \end{aligned}$$

and it is well known that $\phi(E) = \tau(\mathrm{id}_A)$. If $\phi(E)$ is a torsion class it follows from the exactness of (2.3) that $\phi \circ i = \lambda \cdot \mathrm{id}_A$ for some $\lambda \in \boldsymbol{Z}^+$ and $\phi \in H^1(\Gamma', A) = \mathrm{Hom}(\Gamma', A)$ (A is a trivial Γ'-module).

REMARK 2.4. Up to now our considerations apply equally well to the primary obstruction of flat bundles induced by any homomorphism $\alpha: \Gamma \to G$ of a discrete group Γ into a path-connected topological group G. The next lemma, however, will make use of the fact that G is a nilpotent Lie group.

LEMMA 2.5. *Let $\alpha': \Gamma' \to G'$ and $A = \Gamma' \cap Z(G')$, as in* §1. *If there exists a homomorphism $\phi \in \mathrm{Hom}(\Gamma', A)$ such that $\phi \circ i = \lambda \cdot \mathrm{id}_A$, $\lambda \in \boldsymbol{Z}^+$ where $i: A \to \Gamma'$ is the inclusion, then G' is isomorphic to the vector group $\boldsymbol{R}^n$.*

It is clear from what has already been said that Lemma 2.5 will complete the proof of Theorem 1.1.

3. First we list some known facts about 1-connected and connected nilpotent Lie groups and their discrete uniform subgroups. Let $\alpha': \Gamma' \subset G'$ be such a pair and $\mathfrak{g}$ the Lie algebra of G'. Then there exists a base $\xi_1, \cdots, \xi_n$ of $\mathfrak{g}$ such that the map $\rho: \mathfrak{g} \to G'$ defined by $\xi = \sum_{i=1}^n \lambda_i \cdot \xi_i \rightsquigarrow \rho(\xi) = \prod_{i=1}^n \exp(\lambda_i \cdot \xi_i)$ is a homeomorphism and $\rho(\xi) \in \Gamma'$ if and only if $\lambda_i \in \boldsymbol{Z}$ (Malcev coordinates [**6**]). Moreover the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is nontrivial, say of dimension $m > 0$ [**1**; **4**, Corollary 1] and $\exp: \mathfrak{z}(\mathfrak{g}) \to Z(G')$ is a homeomorphism [**7**, Lemma 3]. Hence $Z(G')$ is a vector group $\boldsymbol{R}^m$. We quote the following results as a lemma.

LEMMA 3.1. (i) [**6**] *G' has a discrete uniform subgroup if and only if there exists a (nilpotent) Lie algebra $\mathfrak{h}$ over the rationals $\boldsymbol{Q}$ such that $\mathfrak{g} = \mathfrak{h} \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ (i.e. $\mathfrak{g}$ is rational).*

(ii) [**9**, Lemma 2.1] *Let $H \subset G'$ be a closed connected subgroup, $\Gamma' \subset G'$ a discrete uniform subgroup. Then $\Gamma' \cap H \subset H$ is uniform if the Lie algebra $\mathfrak{h}$ of H is rational.*

Since $Z(G')$ is trivially rational, it follows that $A = \Gamma' \cap Z(G')$ is uniform in $Z(G')$. Hence A is free abelian of rank m and $Z(G')/A$ is a torus $\boldsymbol{T}^m$.

We are now ready to prove Lemma 2.5. Consider the diagram

$$\begin{array}{ccc} G' & \overset{j'}{\leftarrow} & Z(G') \\ \alpha' \uparrow & & \uparrow \beta' \\ \Gamma' & \underset{i}{\overset{\phi}{\rightleftarrows}} & A \end{array} \quad , \qquad \phi \circ i = \lambda \cdot \mathrm{id}_A,\ \lambda \in \boldsymbol{Z}^+.$$

Using Malcev coordinates we define an analytic homomorphism $\Phi: G' \to Z(G')$ satisfying $\beta' \circ \phi = \Phi \circ \alpha'$ and $\Phi \circ j' = \lambda \cdot \mathrm{id}_{Z'}$ by $\Phi(g) = \sum_{i=1}^{n} t_i \cdot d_i$, where $g = \prod_{i=1}^{n} \exp(t_i \cdot \xi_i)$ and $d_i = \phi(\exp(\xi_i))$ (we use additive notation in $Z(G') \cong \boldsymbol{R}^m$ and omit α' and β' from the formulae). Φ extends ϕ: In fact, for $\gamma = \prod_{i=1}^{n} \exp(t_i \cdot \xi_i) \in \Gamma'$, $t_i \in \boldsymbol{Z}$, we have $\Phi(\gamma) = \sum_{i=1}^{n} t_i \cdot d_i = \sum_{i=1}^{n} t_i \cdot \phi(\exp(\xi_i)) = \phi(\gamma)$ since ϕ is a homomorphism. Let $h = \prod_{i=1}^{n} \exp(s_i \cdot \xi_i)$; then we have $gh = \prod_{i=1}^{n} \exp(u_i \cdot \xi_i)$, $u_i = t_i + s_i + q_i(t_1, \cdots, t_{i-1}; s_1, \cdots, s_{i-1})$ where the q_i are polynomials with rational coefficients and integral values for t_i, $s_i \in \boldsymbol{Z}$ [6]. It follows that $\Phi(gh) = \Phi(g) + \Phi(h) + \Sigma$, $\Sigma = \sum_{i=1}^{n} q_i(t_1, \cdots; s_1, \cdots) \cdot d_i$. Since ϕ is a homomorphism, the polynomial function Σ with values in $Z(G') \cong \boldsymbol{R}^m$ vanishes for t_i, $s_i \in \boldsymbol{Z}$. Hence it vanishes identically and Φ is a homomorphism. Finally, $\Phi \circ j': Z(G') \to Z(G')$ extends $\phi \circ i = \lambda \cdot \mathrm{id}_A$ and since $A \subset Z(G')$ is discrete uniform, this extension is unique and given by $\lambda \cdot \mathrm{id}_{Z'}$.

Since $Z(G')$ is a vector group we can define a new homomorphism $\Psi(g) = 1/\lambda \cdot \Phi(g)$, $g \in G'$ which now satisfies $\Psi \circ j = \mathrm{id}_{Z'}$. Hence we have $G' = Z(G') \times G''$ with $G'' = \ker(\Psi)$. G'' is with G' 1-connected, connected and nilpotent. Moreover G'' is centerless and hence trivial by [1;4, Corollary1]. Therefore $n = m$ and $G' = Z(G') = \boldsymbol{R}^n$.

LEMMA 3.2. (i) *j: $\boldsymbol{T}^m = Z(G')/A \subset G = G'/A$ is a maximal compact subgroup of G, $A = \Gamma' \cap Z(G')$.*

(ii) *$\bar{\alpha}$: $\Gamma = \Gamma'/A \subset G'/Z(G')$ is a discrete uniform subgroup and we have a principal fibration*

$$\boldsymbol{T}^m \to G'/\Gamma' \to (G'/Z(G'))/\Gamma. \tag{3.3}$$

PROOF. (i) The fibration $Z(G')/A \to G'/A \to G'/Z(G')$ shows at once that $Z(G')/A$ is maximal compact in G'/A, since $G'/Z(G')$ is contractible. By Lemma 3.1 (ii), $Z(G') \cdot \Gamma'$ is closed in G' and hence Γ is discrete in $G'/Z(G')$. The fact that (3.3) is a fibration follows now easily. Since G'/Γ' is compact, the same must be true for $(G'/Z(G'))/\Gamma$.

REMARK 3.3. By Lemma 3.2, j: $\boldsymbol{T}^m \subset G$ is a homotopy equivalence and so is the map in classifying spaces, Bj: $B_{\boldsymbol{T}^m} \to B_G$. The isomorphisms referred to in Remark 2.2 are those induced by j resp. Bj (we choose standard generators in $\boldsymbol{T}^m$).

4. Let G be a connected Lie group, j: $K \subset G$ a maximal compact subgroup, α: $\Gamma \subset G$ a discrete uniform subgroup and assume for sim-

plicity that $\Gamma \cap Z(G) = 0$ and that Γ is torsion-free. Then Γ acts freely and properly discontinuous on $K\backslash G$, which is topologically a euclidean space; hence the compact manifold $(K\backslash G)/\Gamma$ is of type $K(\Gamma, 1)$ and can be taken as a classifying space B_Γ. With respect to the parallelization of G by right invariant vector fields the tangent bundle of B_Γ is of the form $((\mathfrak{g}/\mathfrak{k} \times_K G)/\Gamma \rightarrow (K\backslash G)/\Gamma$, where K acts on $\mathfrak{g}/\mathfrak{k}$ by the isotropy representation $\rho: K \rightarrow GL^+(\mathfrak{g}/\mathfrak{k})$. It is then easily seen that the composite homomorphism

$$(4.1)\quad H^*(B_{GL^+}, Q) \xrightarrow{B\rho^*} H^*(B_K, Q) \underset{\cong}{\xrightarrow{(Bj^*)^{-1}}} H^*(B_G, Q) \xrightarrow{B\alpha^*} H^*(B_\Gamma, Q)$$

is the characteristic homomorphism of the tangent bundle of B_Γ, so that this tangent bundle is in a weak sense associated to the flat G-bundle $\eta: E_\Gamma \times_\Gamma G \rightarrow B_\Gamma$, $E_\Gamma = K\backslash G$ with classifying map $B\alpha: B_\Gamma \rightarrow B_G$.

Let now Γ and G as in Theorem 1.1. Then by Lemma 3.2 $K = T^m$ and $B_\Gamma = (G'/Z(G'))/\Gamma = (G/T^m)/\Gamma$. Since T^m is in the center of G, it follows that the isotropy representation ρ is trivial and that the tangent bundle of B_Γ is trivial. Looking at (4.1) one sees that the characteristic classes $x_k(\eta) = B\alpha^*(x_k)$, $k = 1, \cdots, m$ of the flat G-bundle η in Corollary 1.2 are *not* tangent classes of B_Γ.

A detailed study of the relation between the characteristic classes of the representation $\alpha: \Gamma \subset G$ and the tangent classes of the manifold $B_\Gamma = (K\backslash G)/\Gamma$ in the more general case described above will be made in a subsequent paper.

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