MORE ON RINGS ON RINGS[1]

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This note is a sequel to the paper *On rings on rings* by Anatole Beck [1]. The problem we consider (originally proposed by Paul Rosenbloom) is that of characterizing the parameter ρ for the annulus $\Omega = \{1 < |z| < \rho\}$ in terms of the ring R of bounded analytic functions on this annulus. Beck's solution involves properties of univalent functions, and although the subset of univalent functions in R can be characterized algebraically, it seems preferable to avoid this complication.

THEOREM. *Let R be the Banach algebra of all bounded analytic functions on the annulus $\Omega = \{1 < |z| < \rho\}$ endowed with the usual* sup norm. *Let U be the set of invertible elements in R (i.e., the set of $f \in R$ for which $1/f \in R$), and let H be the set of $f \in R$ which possess nth roots $f^{1/n} \in R$ for all n. Then*

$$\rho = \inf_{f \in U - H} \|f\| \cdot \|f^{-1}\|. \tag{1}$$

PROOF. The proof is based on a theorem of Schiffer and Huber (cf. [2]):

Let $f: \Omega \to \Omega$ be analytic and map a generator γ for the homology group $H_1(\Omega)$ onto a curve which is homologous to γ^q. Then $q = 0$, 1, or -1, and in the last two cases $f(z)$ is a constant multiple of z or $1/z$ respectively.

To deduce our result, we note first that $f(z) = z$ satisfies $\|f\| \cdot \|f^{-1}\| = \rho$. Now suppose that f is an element of $U - H$ such that $\|f\| \cdot \|f^{-1}\| < \rho$. Multiplying f by a constant, we can adjust the norms so that $\|f\| < \rho$ and $\|f^{-1}\| < 1$. Then f maps the annulus $\{1 < |z| < \rho\}$ into itself. Since $f \notin H$, $f^{1/n}$ fails to exist for some n, and hence if γ is a generator for $H_1(\Omega)$, $f(\gamma)$ cannot be homologous to zero. Thus by the Schiffer-Huber theorem, $f(z)$ is a constant multiple of either z or $1/z$; in either case $\|f\| \cdot \|f^{-1}\| = \rho$.

REFERENCES

1. A. Beck, *On rings on rings*, Proc. Amer. Math Soc. **15** (1964), 350–353.

2. E. Reich, *Elementary proof of a theorem on conformal rigidity*, Proc. Amer. Math. Soc. **17** (1966), 644–645.

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