NONLINEAR EIGENVALUE PROBLEMS AND GALERKIN APPROXIMATIONS

BY FELIX E. BROWDER

Communicated December 21, 1967

Let X be a reflexive Banach space, T and S two mappings of X into its conjugate space X^*. We denote the pairing between w in X^* and u in X by (w, u), and weak convergence (in either X or X^*) by $\rightharpoonup$, strong convergence (in either X or X^*) by $\rightarrow$.

By an eigenvalue problem for the pair (T, S), we mean the problem of finding an element u in X and a real number λ such that

$$T(u) = \lambda S(u), \tag{1}$$

with u possibly satisfying additional normalization conditions. It is our purpose in the present note to describe a way of applying a method of Galerkin type to such problems which works in particular for nonlinear elliptic boundary value problems of variational type. We obtain from it a general theorem on the existence of normalized eigenfunctions for the latter problem, and in the case of T and S odd operators, we obtain also an extremely general form of a theory of Lusternik-Schnirelman type guaranteeing the existence of infinitely many distinct normalized eigenfunctions.

We consider first some restrictions that may be placed on the nonlinear operator T.

DEFINITION 1. *T is said to satisfy condition (S) if for any sequence $\{u_j\}$ in X with $u_j \rightharpoonup u$ in X and $(T(u_j)-T(u), u_j-u)\rightarrow 0$, we have $u_j \rightarrow u$ in X.*

DEFINITION 2. *T is said to satisfy condition $(S)_0$ if for each sequence $\{u_j\}$ in X with $u_j \rightharpoonup u$ in X, $T(u_j) \rightharpoonup z$ in X^*, and $(T(u_j), u_j) \rightarrow (z, u)$, we have $u_j \rightarrow u$ in X.*

LEMMA 1. (a) *If T satisfies condition (S), it satisfies condition $(S)_0$.*

(b) *If T is continuous and satisfies condition $(S)_0$, and if K is any compact set of X^*, B any bounded closed set of X, then $T^{-1}(K)\cap B$ is compact.*

(c) *If T is continuous and satisfies condition $(S)_0$, then the image under T of any bounded closed set B of X is closed in X^*.*

PROOF OF LEMMA 1. PROOF OF (a). Suppose $u_j \rightharpoonup u$, $T(u_j) \rightharpoonup z$, and $(T(u_j), u_j) \rightarrow (z, u)$. Then

$$(T(u_j) - T(u), u_j - u) = (T(u_j), u_j) - (T(u_j), u) - (T(u), u_j - u) \to (z, u) - (z, u) - 0 = 0.$$

Hence by the condition (S), $u_j \to u$. Q.E.D.

PROOF OF (b). Let $\{u_j\}$ be a sequence in $T^{-1}(K) \cap B$. By passing to a subsequence, we may assume that $u_j \rightharpoonup u$ in X, $T(u_j) \to z$ in K. Hence $(T(u_j), u_j) \to (z, u)$ and, by condition $(S)_0$, $u_j \to u$. Hence $u \in B$, and by the continuity of T, $T(u) = z$, i.e. $u \in T^{-1}(K) \cap B$. Q.E.D.

PROOF OF (c). The conclusion of (b) implies that T is a proper continuous map of B into X^*. Hence it is a closed map of B into X^* and $T(B)$ is closed in X^*. Q.E.D.

We now give our principal methodological result.

THEOREM 1. *Let X be a separable reflexive Banach space, T and S two continuous bounded mappings of X into X^* with T satisfying condition $(S)_0$ and S a compact map of X into X^*. Let $\{X_n\}$ be an increasing sequence of finite dimensional subspaces of X whose union is dense in X, B a closed bounded subset of X. Suppose that for each n, there exists an element u_n of $B \cap X_n$ with the property that*

$$j_n^* T(u_n) = \lambda_n j_n^* S(u_n),$$

where j_n is the injection mapping of X_n into X, and j_n^* is the dual projection of X^* onto X_n^*. Suppose that $|\lambda_n|$ is uniformly bounded.

Then there exists an eigenfunction u of the pair (T, S) in B, i.e. $T(u) = \lambda S(u)$, and for any weakly convergent subsequence $u_{n(k)} \rightharpoonup u$ of the sequence $\{u_n\}$, u is such an eigenfunction and $u_{n(k)} \to u$.

PROOF OF THEOREM 1. Since B is bounded and X is reflexive, the sequence $\{u_n\}$ has a weakly convergent subsequence. We may replace the original sequence by this subsequence and assume that $u_n \rightharpoonup u$. It suffices to show that $\{u_n\}$ has a strongly convergent subsequence and that u is an eigenfunction of the pair (T, S). Since $|\lambda_n|$ is uniformly bounded, we may assume for our original sequence (again by passing to an infinite subsequence) that $\lambda_n \to \lambda$, and since S is compact, that $S(u_n) \to w$ in X^*.

Let v be any element of V_m for some m, and consider $n \geqq m$. Then,

$$(Tu_n, v) = (Tu_n, j_n v) = (j_n^* T(u_n), v) = \lambda_n (j_n^* S(u_n), v) = \lambda_n (S(u_n), v).$$

Hence

$$(T(u_n), v) \to \lambda(w, v), \ (n \to +\infty).$$

Since this is true for each v in the dense union of the spaces V_m and since the sequence $\{T(u_n)\}$ is bounded, it follows that $T(u_n) \rightharpoonup \lambda w$.

On the other hand, by the same argument,

$$(T(u_n), u_n) = \lambda_n(S(u_n), w) \to \lambda(w, v).$$

Applying the condition $(S)_0$ for T, we see that $u_n \to u$. Since T and S are continuous, $T(u_n) \to T(u)$, $S(u_n) \to w$. Hence

$$T(u) = \lim_n T(u_n) = \lambda w = -S(u).$$

Q.E.D.

The special interest of the conditions (S) and $(S)_0$ is that they are satisfied by quasi-linear elliptic differential operators in generalized divergence form under extremely weak hypotheses on the operators.

THEOREM 2. *Let Ω be a bounded open set in R^n for which the Sobolev Imbedding Theorem is valid, A and B two differential operators on Ω of the form*

$$A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, Du, \cdots, D^m u),$$

$$B(u) = \sum_{|\beta| \leq m-1} (-1)^{|\beta|} D^\beta B_\beta(x, u, \cdots, D^m u).$$

For each α and β, let $A_\alpha(x, \xi)$ and $B_\beta(x, \xi)$ be continuous in x and Lebesgue measurable in ξ. Suppose that for a given exponent p with $1 < p < +\infty$, V is a closed subspace of the Sobolev space $W^{m,p}(\Omega)$ and for u and v in V, we set

$$a(u, v) = \sum_{|\alpha| \leq m} (A_\alpha(x, u, Du, \cdots, D^m u), D^\alpha v),$$

$$b(u, v) = \sum_{|\beta| \leq m-1} (B_\beta(x, u, Du, \cdots, D^m u), D^\beta v),$$

(with $(w, v) = \int_\Omega wv$). Suppose that the following three conditions are satisfied:

(1) *There exists a constant c_0 and functions c_α in $L^{p'}(\Omega)$ such that*

$$| A_\alpha(x, \xi) | \leqq c_\alpha(x) + c_0 \sum_{|\phi| = m} | \xi_\phi |^{p-1} + \sum_{|\phi| \leq m-1} | \xi_\phi |^{q_\alpha},$$

$$| B_\beta(x, \xi) | \leqq c_\beta(x) + c_0 \sum_{|\phi| = m} | \xi_\phi |^{q_{\beta\phi}},$$

where

$$q_{\alpha\phi} < p_\phi q_\alpha^{-1}, \quad q_\alpha = \max(1, np(np - n + p(m - |\alpha|))^{-1}),$$
$$p_\phi^{-1} = \max(0, np(n - p(m - |\phi|))^{-1}).$$

(2) *For $\psi = \{\psi_\beta : |\beta| \leqq m-1\}$, $\zeta = \{\zeta_\alpha : |\alpha| = m\}$, set $A_\alpha(x, \psi, \zeta)$*

$=A(x, \xi)$ *where* $\xi=[\psi, \zeta]$. *Then for every* x *in* Ω, ψ, ζ *and* ζ' *with* $\zeta \neq \zeta'$,

$$\sum_{|\alpha|=m} [A_\alpha(x, \psi, \zeta) - A_\alpha(x, \psi, \zeta')](\zeta_\alpha - \zeta'_\alpha) > 0.$$

(3) *There exist positive constants* c_1 *and* c_2 *such that*

$$\sum_{|\alpha|\leq m} A_\alpha(x, \xi)\xi_\alpha \geq c_1 |\xi|^p - c_2.$$

Then: (a) *The form* $a(u, v)$ *is well defined for all* u *and* v *in* V *and there exists an unique element* $T(u)$ *in* V^* *such that* $a(u, v)=(T(u), v)$ *for all* v *in* V *and a given element* u *in* V. *Similarly,* $b(u, v)$ *is well defined for* u *and* v *and* $b(u, v)=(S(u), v)$ *for all* v *in* V *and a given* u *in* V, *when* $S(u)\in V^*$.

(b) T *is a bounded continuous mapping of* V *into* V^* *which satisfies condition* (S).

(c) S *is a compact mapping of* V *into* V^*.

The proof of Theorem 2 and the details of further applications of these arguments will be given in another paper.

Let us consider, however, the application of Theorems 1 and 2 to the "self-adjoint" case, i.e. when A and B are the Euler-Lagrange operators of multiple integral variational problems.

THEOREM 3. *Let* T *and* S *be the derivatives of two* C_1 *functions* f *and* g *on* V, *respectively, where* T *is bounded and satisfies condition* $(S)_0$ *and* S *is compact. Let* c *be a constant such that on the level set* $M_c = \{u | f(u)=c\}$, $(T(u), u)>0$, *while* M_c *is bounded. Suppose that* $g(u)>0$ *for* u *in* M_c, *that* $(S(u), u)>0$ *on* M_c, *and that for each set* B *on* M_c *for which* $g(u)>\epsilon>0$, $(S(u), u)>d(\epsilon)>0$.

Then g *assumes its maximum at a point* u_0 *of* M_c, *and* $T(u_0)=\lambda S(u_0)$ *for some* $\lambda>0$.

PROOF OF THEOREM 3. V is assumed as in Theorem 1 to be a separable reflexive Banach space. We choose an increasing sequence V_n of finite dimensional subspaces whose union is dense in V and with $M_c \cap V_n$ having their union dense in M_c. Let f_n and g_n be the restrictions of f and g to V_n. Then $M_c \cap V_n$ is the c-level set of f_n and $f_n' = j_n^*T$, $g_n' = j_n^*S$. Since $(f_n'(u), u) = (T(u), u) > 0$ on $M_c \cap V_n$, $M_c \cap V_n$ is a manifold. The function g is C^1 on this compact manifold and assumes its maximum m_n on $M_c \cap V_n$ at a point u_n which satisfies the condition $T(u_n) = \lambda_n S(u_n)$. Since $g(u_n) = m_n \to m = \sup_{u\in M_c} g(u)$, $(S(u_n), u_n) \geq d_0 > 0$ for all n. Hence, since

$$\lambda_n = (T(u_n), u_n)/(S(u_n), u_n),$$

λ_n is uniformly bounded. If we apply Theorem 1, we obtain the conclusion that for an infinite subsequence, $u_{n(k)} \to u$, where u is an eigenfunction $T(u) = \lambda S(u)$. Since g is continuous, $g(u_{n(k)}) \to g(u) = m$. Since M_c is closed, $u \in M_c$. Q.E.D.

THEOREM 4. *Let V be a separable reflexive Banach space, T and S two continuous mappings of V into V^* with T bounded and satisfying condition $(S)_0$, S compact. Suppose that T and S are the derivatives of two C^1 functions f and g on V, and suppose that on the level set $M_c = \{u \mid f(u) = c\}$, $(T(u), u) > 0$. Suppose that M_c is invariant under the involution $\pi(u) = -u$, and that $g(-u) = g(u)$ on M_c. Suppose further that M_c is intersected exactly once by each ray through the origin, that $g(u) > 0$ for u in M_c, that $(S(u), u) > 0$ on M_c and that $g(u)$ and $(S(u), u)$ go to zero together on M_c. Suppose finally that for each $\epsilon > 0$, there exists a finite dimensional subspace V_ϵ of V such that outside the ϵ-neighborhood of V_ϵ, $g(u) < \epsilon$. For each j, let*

$$h_j = \sup_{p\text{-cat}(K, M_c) \geq j} \min_{u \in K} g(u),$$

where the supremum is taken over compact subsets K of M_c whose image in M_c/π has Lusternik-Schnirelman category $\geq j$.

Then:

(a) *For each j, h_j is well defined and there exists u_j in M_c with*

$$T(u_j) = \lambda_j S(u_j),\ (\lambda_j > 0), \qquad f(u_j) = c, \qquad g(u_j) = h_j,$$

while $\lambda_j \to +\infty$, $h_j \to 0$.

(b) *Suppose that* $\dim(V_n) \geq j$. *Then we can define*

$$h_{j,n} = \sup_{p\text{-cat}(K, M_c) \geq j, K \subset V_n} \min_{u \in K} g(u),$$

and for each $j \leq n$, there exists $u_{j,n}$ in V_n such that

$$j_n^* T(u_{j,n}) = j_n^* S(u_{j,n}), \qquad f(u_{j,n}) = c, \qquad g(u_{j,n}) = h_{j,n}.$$

(c) *For any fixed j and any infinite subsequence $u_{j,n(k)} \to u_j$ as $k \to \infty$, u_j is an eigenfunction satifsying the condition of part* (a) *and $u_{j,n(k)} \to u_j$.*

PROOF OF THEOREM 4. Since $f_n' = j_n^* T$, so that $(f_n'(u), u) > 0$ on $M_c \cap V_n$, the latter is a manifold for each n, and $(M_c \cap V_n)/\pi$ is homeomorphic to P^{n-1}, which has Lusternik-Schnirelman category n. The conclusions of (b) then follow from the classical Lusternik-Schnirelman theory on finite dimensional manifolds (Lusternik [7], Vainberg [8]). The conclusion of (a) will follow from that of part (c) so that it suffices to prove (c).

PROOF OF (c). We may assume without loss of generality that $u_{j,n} \rightharpoonup u_j$ as $n \to \infty$. Since $g(u_{j,n}) = h_{j,n} \to h_j$ as $j \to +\infty$ where $h_j > 0$ for each j, it follows that $(S(u_{j,n}), u_{j,n}) \geqq d_0 > 0$ for all n. Hence $\lambda_{j,n} = (T(u_{j,n}), u_{j,n})(S(u_{j,n}), u_{j,n})^{-1}$ is uniformly bounded. Applying Theorem 1, we find that $u_{j,n} \to u_j$. Hence $f(u_j) = \lim_n f(u_{j,n}) = c$. Since $g(u_j) = \lim_n g(u_{j,n}) = h_j$, and since by Theorem 1, u_j is an eigenfunction of the pair (T, S), our conclusion follows. Q.E.D.

REMARKS. (1) The result of Theorem 4 combined with Theorem 2 generalizes the writer's results in [4] under weaker regularity and boundedness hypotheses on the A_α and makes no explicit use of the theory of infinite dimensional manifolds.

(2) An earlier attempt to weaken the regularity hypotheses of [4] was made by M. Berger [1] using an infinite dimensional argument. His argument in [1] contains a number of serious errors and gaps which make it doubtful that the argument can be carried through (cf. the review by C. W. Clark in Math. Reviews).

(3) A recent paper with a similar title by S. Hildebrandt [6] has no intersection with the present paper since it concerns linear operators depending nonlinearly on λ, not nonlinear operators depending linearly on λ. However, the methods of the present paper can be used to combine Hildebrandt's results with those given here and extend them to nonlinear operators.

BIBLIOGRAPHY

1. M. Berger, *A Sturm-Liouville Theorem for nonlinear elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa **20** (1966), 543–582.

2. F. E. Browder, *Nonlinear elliptic boundary value problems*, Bull. Amer. Math. Soc. **69** (1963), 862–874.

3. ———, *Variational methods for nonlinear elliptic eigenvalue problems*, Bull. Amer. Math. Soc. **71** (1965), 176–183.

4. ———, *Infinite dimensional manifolds and nonlinear elliptic eigenvalue problems*, Ann. of Math. **82** (1965), 459–477.

5. ———, *Problèmes nonlinéaires*, University of Montreal Press, Montreal, 1966.

6. S. Hildebrandt, *Über die Lösung nichtlinearer Eigenwertaufgaben mit der Gälerkinverfahren*, Math. Z. **101** (1967), 255–264.

7. L. A. Lusternik, *The topology of the calculus of variations in the large*, Transl. Math. Monographs, vol. **16**, Amer. Math. Soc., Providence, R. I., 1966.

8. M. M. Vainberg, *Variational methods for the study of nonlinear operators*, Holden-Day, San Francisco, 1964.

UNIVERSITY OF CHICAGO