THE TOPOLOGICAL DEGREE AND GALERKIN APPROXIMATIONS FOR NONCOMPACT OPERATORS IN BANACH SPACES

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Let X and Y be real Banach spaces, G a bounded open subset of X, $\text{cl}(G)$ its closure in X, $\text{bdry}(G)$ its boundary in X. We consider mappings T, (nonlinear, in general), of $\text{cl}(G)$ into Y which are *A-proper*, in the sense defined below, with respect to a given approximation scheme of generalized Galerkin type. We define a generalized concept of *topological degree* for such mappings with respect to the given approximation scheme, and show that this degree (which may be multivalued) has the basic properties of the classical Leray-Schauder degree (where the latter is defined on the narrower class of maps of X into X of the form $I+C$, with I the identity and C compact).

For a wide class of A-proper mappings T of the form $T=H+C$, with H an A-proper homeomorphism of a suitable type and C compact, we show that the degree is single-valued and coincides with another generalized degree studied in Browder [9] and Browder-Nussbaum [11], and in particular is independent of the approximation scheme involved. In particular, this holds if H is *strongly accretive* from X to X (cf. Browder [4], [5], [6], [8]), including as a very special case all maps H of the form $H=I-U$, with U a strict contraction.

DEFINITION 1. *Let X and Y be Banach spaces. By an (oriented) approximation scheme for mappings from X to Y, we mean: an increasing sequence $\{X_n\}$ of oriented finite dimensional subspaces of X, an increasing sequence $\{Y_n\}$ of oriented finite dimensional subspaces of Y, and a sequence of linear projection maps $\{Q_n\}$ with Q_n mapping Y on Y_n such that* $\dim(X_n)=\dim(Y_n)$ *for all n, $\bigcup_n X_n$ is dense in X, and $Q_n y\to y$ as $n\to\infty$ for all y in Y.*

DEFINITION 2. *Let G be a bounded open subset of X, T a mapping of* $\text{cl}(G)$ *into Y. Then T is said to be A-proper with respect to a given approximation scheme in the sense of Definition* 1 *if for any sequence $\{n_j\}$ of positive integers with $n_j\to\infty$ and a corresponding sequence $\{x_{n_j}\}$ in* $\text{cl}(G)$ *with each x_{n_j} in X_{n_j} such that $Q_{n_j}Tx_{n_j}$ converges strongly in Y to an element y, there exists an infinite subsequence $\{n_{j(k)}\}$ such that $x_{n_{j(k)}}$ converges strongly to x in X as $k\to\infty$ and $T(x)=y$.*

The concept of A-proper mapping is a slight variant of the condition (H) of Petryshyn [18], and both are modifications of the definition of *P-compact* mapping in Petryshyn [15], [16], and [17]. A similar definition has been given for *strongly closed* mappings by Pohojhayev [20].

DEFINITION 3. *Let T be an A-proper continuous mapping from* $\mathrm{cl}(G)$ *to Y with respect to a given approximation scheme, and let a be a point of* $Y-T(\mathrm{bdry}(G))$. *Let* $G_n=G\cap X_n$, *and let* $T_n=Q_nT|_{G_n}$.

We define $\mathrm{Deg}(T, G, a)$, *the degree of T on G over a with respect to the given scheme, as follows: Let Z' be the set of all integers (positive, negative, and zero) together with* $\{+\infty\}$ *and* $\{-\infty\}$. *Then* $\mathrm{Deg}(T, G, a)$ *is the subset of Z' given by*:

(1). *The integer m lies in* $\mathrm{Deg}(T, G, a)$ *if there exists an infinite sequence of positive integers n such that* $\deg(T_n, G_n, Q_na)$ *is well-defined and equals m.*

(2). $\pm\infty$ *lies in* $\mathrm{Deg}(T, G, a)$ *if there exists an infinite sequence of integers* $\{n_j\}$ *with* $n_j\to\infty$ *such that* $\deg(T_{n_j}, G_{n_j}, Q_{n_j}a)$ *is well-defined for each j and* $\deg(T_{n_j}, G_{n_j}, Q_{n_j}a)\to\pm\infty$ *as* $j\to\infty$.

(The degree $\deg(T_n, G_n, Q_na)$ used in Definition 3 is the classical Brouwer degree for mappings of oriented finite dimensional Euclidean spaces of the same dimension.)

Using the properties of the Brouwer degree and of A-proper maps, we obtain a direct and simple proof of the following theorem:

THEOREM 1. *Let X and Y be Banach spaces, G a bounded open subset of X, T an A-proper continuous mapping of* $\mathrm{cl}(G)$ *into Y with respect to a given approximation scheme. Let a be a point of* $Y-T(\mathrm{bdry}(G))$, *and let* $G_n=G\cap X_n$, $T_n=Q_nT|_{G_n}$. *Then*:

(a) *There exists an integer* $n_0\geqq 1$ *such that for* $n\geqq n_0$, Q_na *does not lie in* $T_n(\mathrm{bdry}\ G_n)$. *Hence for such n*, $\deg(T_n, G_n, Q_na)$ *is well-defined, and in particular,* $\mathrm{Deg}(T, G, a)$ *is a nonempty subset of Z'.*

(b) *If* $\mathrm{Deg}(T, G, a)\neq\{0\}$, *there exists an element x of G such that* $T(x)=a$.

(c) *Let T be a continuous mapping of* $\mathrm{cl}(G)\times[0, 1]$ *into Y, and for each t, let* $T_t(x)=T(x, t)$. *Suppose that T is uniformly continuous in t on* $[0, 1]$, *and that for each t, T_t is A-proper with respect to a fixed approximation scheme from X to Y. Then if a lies in* $Y-T(\mathrm{bdry}(G)\times[0, 1])$, *it follows that* $\mathrm{Deg}(T_t, G, a)$ *is independent of t in* $[0, 1]$.

(d) *Let* $G=G_1\cup G_2$, *and for* $G'=(G_1\cap G_2)\cup\mathrm{bdry}(G_1)\cup\mathrm{bdry}(G_2)$, *suppose that $T(G')$ does not contain a. Then*

$$\mathrm{Deg}(T, G, a)\subset\mathrm{Deg}(T, G_1, a)+\mathrm{Deg}(T, G_2, a),$$

with equality holding if either $\mathrm{Deg}(T, G_1, a)$ *or* $\mathrm{Deg}(T, G_2, a)$ *is a singleton integer. (We use the convention that* $\infty-\infty=Z'$.)

Theorem 1 has as corollaries a number of interesting fixed point and mapping theorems for A-proper mappings. In the present discussion, we focus on an important special case for which the degree as defined in Definition 3 is single-valued.

THEOREM 2. *Let X and Y be Banach spaces, G a bounded open subset of X, T a continuous A-proper mapping of* $\mathrm{cl}(G)$ *into Y, $a \in Y - T(\mathrm{bdry}\ G)$. Suppose that we are given an approximation scheme in the sense of Definition* 1 *and $T = H + C$, where C maps* $\mathrm{cl}(G)$ *into a relatively compact subset of Y and H maps G homeomorphically onto an open subset $H(G)$ of Y, carrying* $\mathrm{cl}(G)$ *homeomorphically onto* $\mathrm{cl}(H(G))$. *Let $H_n = Q_n H$, $C_n = Q_n C$, $T_n = H_n + C_n$, with all these mappings restricted to* $\mathrm{cl}(G_n)$ *where $G_n = G \cap X_n$. Suppose that for each n, H_n is an orientation preserving homeomorphism of G_n into Y_n and that the following condition holds:*

(c) *There exists a continuous, strictly increasing function $\alpha(r)$ for $r \geqq 0$ with $\alpha(0) = 0$ such that for all n and each pair u and v in* $\mathrm{cl}(G_n)$,

$$\|H_n(u) - H_n(v)\| \geqq \alpha(\|u - v\|).$$

Then there exists $n_1 \geqq 1$ such that for $n \geqq n_1$,

$$\deg(T_n, G_n, Q_n a) = \deg(I + CH^{-1}, H(G), a).$$

In particular, $\mathrm{Deg}(T, G, a) = \{\deg(I + CH^{-1}, H(G), a)\}$.

COROLLARY TO THEOREM 2. *The conclusion of Theorem 2 holds in the case in which $X = Y$ and $T = H + C$, with C compact and H strongly accretive on X, i.e.*

$$(Hu - Hv, J(u - v)) \geqq c(\|u - v\|)$$

for a continuous strictly increasing function $c(r)$ for $r \geqq 0$ with $c(0) = 0$, and J a duality mapping of X into X^* satisfying the conditions $(Ju, u) = \|Ju\| \cdot \|u\|$ and $\|Ju\| = \psi(\|u\|)$ for a continuous strictly increasing function ψ with $\psi(0) = 0$. We must assume in addition that the family of projections Q_n has $\|Q_n\| = 1$ for all n, and that either X^* is uniformly convex or that H is uniformly continuous on bounded subsets of X. (The latter case includes $H = I - U$, with U a strict contraction.)

PROOF OF THEOREM 2. Since T is A-proper, we may assume that for all n, $Q_n a$ does not lie in $T_n(\mathrm{bdry}\ G_n)$ so that $\deg(T_n, G_n, Q_n a)$ is well-defined. Since H_n is an orientation preserving homeomorphism of G_n into Y_n, we have: $\deg(T_n, G_n, Q_n a) = \deg(I + C_n H_n^{-1}, H_n(G_n), Q_n a)$.

LEMMA 1. *There exists $d > 0$ such that for $n \geqq n_2$, with n_2 sufficiently large, $\|T_n u - Q_n a\| \geqq d$ for all u in* $\mathrm{bdry}(G_n)$. *Hence we may replace the compact map C by $Q_n C$ and the point a by $Q_m a$ for a sufficiently large integer m without changing either of the degrees in the conclusion of Theorem 2 for $n \geqq n_2$.*

PROOF OF LEMMA 1. The second assertion follows from the first, along with standard properties of the degree. Suppose that the first assertion is false. Then there exists a sequence $\{n_j\}$ with $n_j \to \infty$ and a sequence $\{u_{n_j}\}$ with $u_{n_j} \in \text{bdry}(G_{n_j})$ such that $\|T_{n_j}(u_{n_j}) - Q_{n_j}(a)\| \to 0$. Since T is A-proper, we may assume that u_{n_j} converges strongly to u in X and that $Tu = a$. Since each u_{n_j} lies in bdry(G), u must lie in bdry(G). By hypothesis, there are no points in bdry(G) for which $Tu = a$. q.e.d.

LEMMA 2. *Let U be any neighborhood in $H(G)$ of the set $K_1 = \{v \mid v \in H(G),\ v = H(u),\ \text{where } T(u) = a\}$. Let S_n be the mapping of $H_n(G_n)$ into Y given by $S_n u = Q_n C H_n^{-1}(u)$. Then there exists $n_3 \geqq 1$ such that for $n \geqq n_3$, any point v_n in $H_n(G_n)$ such that $(I + S_n)(v_n) = Q_n a$ must lie in the given neighborhood U.*

PROOF OF LEMMA 2. Suppose not. Then there will exist an infinite sequence $\{v_{n_j}\}$ with $v_{n_j} \in H_{n_j}(G_{n_j})$ and $n_{j_1} \to \infty$ such that $v_{n_j} + S_{n_j} v_{n_j} = Q_{n_j} a$ with each v_{n_j} outside of U. Let $z_{n_j} = H_{n_j} v_{n_j}$, $z_{n_j} \in G_{n_j}$. Then

$$T_{n_j}(z_{n_j}) = v_{n_j} + C_{n_j} \overset{-1}{H_{n_j}} v_{n_j} = Q_{n_j} a.$$

Since T is A-proper, we may pass to an infinite subsequence and assume that z_{n_j} converges strongly to an element z of G for which $Tz = a$. Hence $v_{n_j} = Q_{n_j} H(z_{n_j})$ converges strongly to $H(z)$, which lies in K_1. Since U is a neighborhood of K_1, this contradicts the fact that all the v_{n_j} lie outside of U. q.e.d.

LEMMA 3. *The set K_1 defined in Lemma 2 is compact, and there exists a neighborhood U_1 of K_1 and an integer n_4 such that for $n \geqq n_4$, U_1 is contained in $Q_n^{-1}(H_n(G_n))$.*

PROOF OF LEMMA 3. The compactness of K_1 follows easily from the fact that T is A-proper. Suppose the remainder of the assertion of Lemma 3 were not true. Then there would exist a sequence $\{y_{n_j}\}$ for $n_j \to \infty$ such that dist$(y_{n_j}, K_1) \to 0$ for which $Q_{n_j} y_{n_j}$ does not lie in $Q_{n_j}(H(G \cap X_{n_j}))$. Since K_1 is compact, we may assume that y_{n_j} converges strongly as $j \to \infty$ to an element y of K_1. Since $K_1 \subset H(G)$, we may assume that each y_{n_j} lies in $H(G)$ and form $w_{n_j} = H^{-1}(y_{n_j})$. By the continuity of H^{-1}, $w_{n_j} \to w$ where $H(w) = y$, and $T(w) = a$.

For each n, we set $\epsilon_n = 2$ dist$(K_1, H(G_n))$. Then $\epsilon_n \to 0$ as $n \to \infty$, and for each y_{n_j} in the preceding paragraph, we may find u_{n_j} in G_{n_j} such that $\|y_{n_j} - H(u_{n_j})\| \leqq \epsilon_{n_j}$. The hypothesis of Theorem 2 implies that there exists a constant $c > 0$ such that $\|Q_n\| \leqq c$ for all n. Hence, $\|Q_{n_j} y_{n_j} - Q_{n_j} H(u_{n_j})\| \leqq c\epsilon_{n_j}$. Since bdry$(H_n(G_n)) = H_n(\text{bdry } G_n)$ for all n, it follows that dist$(y_{n_j}, H_{n_j}(\text{bdry } G_{n_j})) \leqq c\epsilon_{n_j}$. Hence, we may find elements v_{n_j} in bdry G_{n_j} such that $\|Q_{n_j}y_{n_j}-Q_{n_j}H(v_{n_j})\|\to 0$. Since T is A-proper, so is $H=T-C$. Passing to an infinite subsequence, we may assume that v_{n_j} converges strongly to an element v of bdry G for which $H(v)=y$, i.e. $v=w$ and $T(v)=a$. This is a contradiction, proving the lemma. q.e.d.

PROOF OF THEOREM 2 COMPLETED. By Lemma 1, we may assume that for $n\geqq n_3$, $Q_nC=C$ and $Q_na=a$. We know that

$$\deg(T_n,\, G_n,\, Q_na) = \deg(I + C_nH_n^{-1},\, H_n(G_n),\, Q_na),$$

and that

$$d_n = \deg(I + C_nH_n^{-1}, H_n(G_n), Q_na) = \deg(I + CH_n^{-1}Q_n, Q_n^{-1}(H_n(G_n)), a).$$

We wish to show this last degree to be equal to

$$\delta = \deg(I + CH^{-1},\, H(G),\, Q_na) = \deg(I + CH^{-1},\, H(G),\, a).$$

By Lemmas 2 and 3, we may choose a neighborhood U of K_1 in $H(G)$ such that: $U\subset Q_n^{-1}(H_n(G_n))$, while for any v_n in $H_n(G_n)$ such that $(I+C_nH_n^{-1})(v_n)=Q_na$, we have $v_n\in U$. By Lemma 1, we may assume that $a=Q_ma$, $C=Q_mC$. Hence for any v in $Q_n^{-1}(H_n(G_n))$ such that $v+CH_n^{-1}Q_nv=a$, we have $v\in Y_n$ and $Q_nv=v$ so that v lies in $H_n(G_n)$. Thus,

$$d_n = \deg(I + CH_n^{-1},\, U\cap Y_m,\, a); \qquad \delta = \deg(I + CH^{-1},\, U\cap Y_m,\, a).$$

It suffices by the properties of the degree (e.g. [14]) to show that the mappings CH_n^{-1} converge uniformly to the mapping CH^{-1} on the compact set K_3 which is the closure of $U\cap Y_m$ in $H(G)$.

Let $u\in K_3$, and set $w=CH^{-1}(u)$, $w_n=CH_n^{-1}(u)$, $x=H^{-1}(u)$, $x_n=H_n^{-1}(u)$. The set $K_4=H^{-1}(K_3)$ is a compact subset of G, and hence $\operatorname{dist}(K_4,\, G_n)=2\epsilon_n\to 0$. Therefore, we may find y_n in G_n such that $\|x-y_n\|\leqq\epsilon_n$. Since every continuous mapping is uniformly continuous at the points of a compact subset of its domain, there exists a sequence $\beta_n\to 0$ such that for all u in K_3 and the corresponding point x, $\|H(x)-H(y_n)\|\leqq\beta_n$. Since $H(x)=u$ and $\|Q_n\|\leqq c$, we have

$$\|H_n(x_n) - H_n(y_n)\| = \|u - Q_nH(y_n)\| \leqq \|u - Q_nu\| + \beta_n.$$

Since K_3 is compact, there exists $\zeta_n\to 0$ such that $\|u-Q_nu\|\leqq\zeta_n$ for u in K_3. Applying the condition (c) of the hypothesis of Theorem 2, we obtain

$$\alpha(\|x_n - y_n\|) \leqq \|H_n(x_n) - H_n(y_n)\| \leqq \beta_n + \zeta_n,$$

so that

$$\|x - x_n\| \leqq \|x - y_n\| + \|y_n - x_n\| \leqq \epsilon_n + \alpha^{-1}(\beta_n + \zeta_n) \to 0,$$

so that $H_n^{-1}u$ converge uniformly to $H^{-1}u$ on K_3. Finally, C is continuous from cl(G) to Y and hence uniformly continuous at points of the compact set K_4. Hence $CH_n^{-1}(u)$ converges uniformly to $CH^{-1}(u)$ for u in K_3. q.e.d.

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