DIFFRACTION BY A HYPERBOLIC CYLINDER

BY CLIFFORD O. BLOOM[1]

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We investigate the asymptotic behavior as $K\to\infty$ of the solution $G(x, y; x_0, y_0: K)$ of the following scattering problem P:

(i) $[\Delta+K^2]U=\delta(x-x_0, y-y_0), \quad (x, y), (x_0, y_0)\in D;$

(ii) $\partial_n U=0, \quad (x, y)\in C;$

(iii) $\lim_{\rho\to\infty} \int_{\Sigma(\rho,0)\cap\bar{D}} |\partial U/\partial r-iKU|^2 dS=0.$

Here C is the left branch of the coordinate hyperbola $(x/h\cdot\cos \mathfrak{n})^2 - (y/h\cdot\sin \mathfrak{n})^2=1$, $\pi/2<\mathfrak{n}<\pi$. In parametric form C is given by the equations $x=h\cdot\cos \mathfrak{n}\cdot\cosh \xi$, $y=\pm h\cdot\sin \mathfrak{n}\cdot\sinh \xi$, $\xi\geqq 0$.

D is the infinite two dimensional region bounded by the convex side of C; D consists of all points (x, y) with elliptic coordinates (ξ, η) such that $\xi\geqq 0$, and $-\mathfrak{n}\leqq\eta\leqq\mathfrak{n}$. $\bar{D}=D\cup C$, and $\Sigma(\rho, 0)=\{(x, y): x^2+y^2=\rho^2\}$.

Δ is the two dimensional Laplacian, $\delta(x-x_0, y-y_0)$ is Dirac's δ-function, and ∂_n denotes differentiation in the direction of the outward normal to C.

Our result is a rigorous asymptotic expansion of the Green's function G as $K(>0)\to\infty$ that holds uniformly in every closed bounded subset $S_<(x_0, y_0)$ of the "shadow" $S(x_0, y_0)$ of C. $S(x_0, y_0)$ consists of those points in $D\cup C$ that cannot be joined to (x_0, y_0) by a line segment lying entirely in $D\cup C$.

The asymptotic expansion we get for G confirms the geometrical theory of diffraction by convex cylinders of infinite cross section (see [1]).

Furthermore, our rigorous asymptotic solution of the problem P can be used with certain bounds to obtain asymptotic solutions of a general class of scattering problems with smooth convex boundaries C' that coincide with C in neighborhoods of the points of "diffraction"; the points where the boundary of $S(x_0, y_0)$ intersects C. For example if C' is formed by joining a convex arc A to the "illuminated" part of C, then an asymptotic expansion of the solution G' in the shadow $S'(x_0, y_0)$ $(=S(x_0, y_0))$ can be obtained once it is known, for some positive N, that $G'(x, y; x_0, y_0: K)=O(K^N)$ as $K\to\infty$, uniformly in (x, y), $(x, y)\in A$.

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The following is an outline of the argument we use to obtain an asymptotic solution of the problem P. The details of the argument are given in [2].

We first express the problem in elliptic coordinates, and then construct a Fourier expansion of G with respect to the eigenfunctions $P^{(j)}(\eta;\tau_n^{(j)},K)$, $j=1,2$, $n=1,2,3,\cdots$ of the operator $L_\eta=d^2/d\eta^2-(Kh\cdot\cos\eta)^2$. $P^{(1)}(\eta;\tau,K)$ and $P^{(2)}(\eta;\tau,K)$ are the first and second principal solutions, relative to $\eta=0$, of Mathieu's equation $(L_\eta+\tau^2)U=0$ on the interval $-\mathfrak{n}\leqq\eta\leqq\mathfrak{n}$. The eigenvalues $\tau_n^{(j)}$ are the positive zeros of the entire function $dP^{(j)}(\mathfrak{n};\tau,K)/d\eta$.

Next we sum the Fourier series to a contour integral $\int_C I(\xi,\eta;\xi_0,\eta_0:Z,K)dZ/2\pi i$, where the integrand I is a meromorphic function of Z, and C is a horizontal line in the upper half of the Z-plane.

For sufficiently large K the poles of I in the upper half of the Z-plane are the zeros $Z_n(K)$ of the entire function $F(\mathfrak{n};Z,K)$ defined as follows. Let $h^{(1)}(\xi;Z,K)$ be an entire function of Z that satisfies the associated Mathieu equation $d^2U/d\xi^2+[(Kh\cdot\cosh\xi)^2-(KZ)^2]U=0$ on $\xi\geqq 0$, and that is asymptotic to $\exp[iKh\cdot\sinh\xi]/(h\cdot\cosh\xi)^{1/2}$ as $\xi\to\infty$. Let $H(\eta;Z,K)$ be the solution of Mathieu's equation on $-\mathfrak{n}\leqq\eta\leqq\mathfrak{n}$ such that $H(0;Z,K)=h^{(1)}(0;Z,K)$ and $dH(0;Z,K)/d\eta=idh^{(1)}(0;Z,K)/d\xi$. Then $F=dH(\mathfrak{n};Z,K)/d\eta$.

The horizontal line C lies below all of the $Z_n(K)$ as $K\to\infty$.

We show that for every positive integer N the contour integral is equal to the sum of the residues $R_n(\xi,\eta;\xi_0,\eta_0:K)$, $n=1,2,\cdots,N$, of I at the poles $Z_n(K)$, and a remainder $\int_{C'} I_N(\xi,\eta;\xi_0,\eta_0:Z,K)dZ$. Here C' is the contour in the 4th quadrant of the Z-plane consisting of (i) the line $\operatorname{Im} Z=\operatorname{Im} Z_{N+1}$ from $\operatorname{Re} Z=-\infty$ to $\operatorname{Re} Z=\operatorname{Re} Z_{N+1}-|Z_{N+1}-Z_N|/2$, (ii) the lower half of the circle $|Z-Z_{N+1}|=|Z_{N+1}-Z_N|/2$, (iii) the line $\operatorname{Im} Z=\operatorname{Im} Z_{N+1}$ from $\operatorname{Re} Z=\operatorname{Re} Z_{N+1}+|Z_{N+1}-Z_N|/2$ to $\operatorname{Re} Z=-\epsilon$, $\epsilon>0$, (iv) the line $\operatorname{Re} Z=-\epsilon$ from $\operatorname{Im} Z=\operatorname{Im} Z_{N+1}$ to $\operatorname{Im} Z=+\infty$.

We prove that $\int_{C'} I_N dZ=O(R_{N+1}(\xi,\eta;\xi_0,\eta_0:K))$ as $K\to\infty$, uniformly in ξ and η, for all (ξ,η) such that $(x,y)\in S_<(x_0,y_0)$.

Finally, we establish that if $(x,y)\in S_<(x_0,y_0)-C$, or if $(x,y)\in S_<(x_0,y_0)\cap C$ the leading term of the asymptotic expansion of R_n is the nth diffraction mode of the geometrical theory of diffraction.

In conclusion we remark that if (x,y) and (x_0,y_0) both lie on C the series $\sum_1^\infty R_n$ converges to G. Otherwise, this series does not represent G, in fact it does not converge. This is in contrast to scattering by an ellipse (see [3] and [4]), a sphere (see [5]), and a parabola (see [6]). For in each of these cases there is a series representation of the solution that is also an asymptotic expansion in the shadow.

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COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY