SOME RESULTS ON ONE-RELATOR GROUPS

BY B. B. NEWMAN

Communicated by L. Auslander, January 11, 1968

The results in this note arose from considering the question: What are the abelian subgroups of a one-relator group? The additive group of p-adic rationals and the free abelian group of rank 2 are certainly subgroups of a one-relator group. For example in

$$G = \text{gp}\{a, b; a^{-1}b^{2p}ab^{-2}\}$$

the infinitely generated subgroup

$$H = \text{sgp}\{b^2, a^{-1}b^2a, a^{-2}b^2a^2, \cdots\}$$

is isomorphic to the additive group of p-adic rationals, and

$$K = \text{sgp}\{b^2, b^{-1}a^{-1}ba\}$$

is free abelian of rank 2. In 1964 Gilbert Baumslag [1] conjectured that the additive group of rationals is not a subgroup of a one-relator group. That this conjecture is correct follows from the following theorem.

THEOREM 1. *Let G be a torsion-free one-relator group. Then no nontrivial element of G has more than finitely many prime divisors. Moreover a nontrivial element is not divisible by more than finitely many powers of a prime p if p is greater than the length of the relator.*

REMARK. An element g of a group is divisible by an integer n, or has a divisor n, if g has an nth root in the group. By the length of the relator is meant the letter length of the relator as a word in a free group.

R. C. Lyndon [7] has shown that the cohomological dimension of a torsion-free one-relator group is $\leqq 2$. Now the cohomological dimension of a free abelian group of rank n is n, and of a direct product of an infinite cyclic group with a noncyclic locally cyclic group is >2, (see [6], [2]). Since the cohomological dimension of a subgroup is less than or equal to the cohomological dimension of the group, it follows that the only abelian subgroups of a torsion-free one-relator group are free abelian of rank $\leqq 2$ or locally cyclic subgroups in which every nontrivial element is divisible by at most finitely many primes.

The proof of Theorem 1 uses the usual argument of the Freiheitssatz (see [9]) together with the following ideas.

DEFINITION. Let H be a subgroup of G and p a prime. Then H is

p-pure in G if for each $g \in G$ and positive integer r, $g^{p^r} \in H$ implies that there exists an element $h \in H$ with $g^{p^r} = h^{p^r}$.

LEMMA 1. *Let $C = \{A * B; J\}$ be the generalized free product of two groups A, B amalgamating a subgroup J. If J is a p-pure subgroup of the factors A, B then A, B are p-pure subgroups of C. Moreover a nontrivial element of C is divisible by all powers of the prime p only if a nontrivial element of A or B is divisible by all powers of the prime p.*

The concept of a p-pure subgroup is the appropriate tool for proving Theorem 1, and indeed the whole argument turns on the following key lemma.

LEMMA 2. *Let G be a one-relator group. Then any subset of the generators of G generates a p-pure subgroup of G where p is any prime greater than the length of the relator.*

For one-relator groups with elements of finite order one can say much more.

THEOREM 2. *Let G be a one-relator group with torsion. Then the centralizer of every nontrivial element of G is cyclic.*

The proof of Theorem 2 is similar to the proof of Theorem 1 except instead of using p-pure subgroups one uses μ-subgroups defined as follows:

DEFINITION. Let H be a subgroup of G. Then H is a μ-subgroup of G if for all $g \in G$,

$$g^{-1}Hg \cap H \neq 1 \quad \text{implies} \quad g \in H.$$

LEMMA 3. *Let $C = \{A * B; J\}$ where J is a μ-subgroup of the factors A, B. Then A, B are μ-subgroups of C. If in A, B the centralizer of every nontrivial element is cyclic, then in C the centralizer of every nontrivial element is cyclic.*

LEMMA 4. *Let G be a one-relator group with torsion. Then any subset of the generators of G generates a μ-subgroup of G.*

The proof of Lemma 4 is not easy and depends on the following seemingly powerful result.

THEOREM 3. *Let $G = \mathrm{gp}\{a, b, \cdots; R^n\}$, $n > 1$, where R is cyclically reduced. Suppose that two words $W = W(a, b, \cdots)$, $V = V(b, \cdots)$, where W is a freely reduced word containing a nontrivially and V does not contain a, define the same element of G. Then W contains a subword which is identical with a subword of $R^{\pm n}$ of length greater than $(n-1)/n$ times the length of R^n.*

This result tells us something about the actual spelling of words representing elements of G.

COROLLARY 1. *The word problem and the extended word problem are solvable in* G.

The proof of this result by W. Magnus (see [9]) is a rather complicated process and it does not show that in the special case of less than $\frac{1}{6}$ groups (see [3]) a much simpler solution is possible. Theorem 3, however, provides the simplest of algorithms.

COROLLARY 2. *Let* F *be a free group on a set* X *of generators and let* $r\in F$ *and* $N=\{r^n\}^F$, $n>1$. *Let* G *and* H *be generated by subsets* Y *and* Z *of* X. *Then* GHN *is a recursive subset of* F.

This solves a problem of R. C. Lyndon (see [8, Problem 3.6]) in the case where N is the normal closure in F of a proper power. Lyndon points out that a solution of this problem enables one to considerably extend the Magnus solution of the word problem for one-relator groups.

The Freiheitssatz for groups with torsion is an immediate consequence of Theorem 3, and may be generalized in several directions, for example:

COROLLARY 3. *Let* $H=\text{sgp}\{a^\beta, b, \cdots\}$ *be a subgroup of* G *where* a, b *are nontrivial in* R. *Then* H *is a free group freely generated by* a^β, $b, \cdots$ *if* β *is any integer* $>2\alpha$ *where* α *is the largest absolute value of an* a*-exponent in* R^n.

This extends a result of N. S. Mendelsohn and Rimhak Ree [10]. If $n\geqq 8$ then the group is a less than $\frac{1}{8}$ group and from results of M. Greendlinger [4] the conjugacy problem is solvable in G if $n\geqq 8$. In this direction see also M. Greendlinger [5] for less than $\frac{1}{6}$ groups, and V. V. Soldatova [11] for a subclass of less than $\frac{1}{4}$ groups. By using a generalization of μ-subgroups one may prove

THEOREM 4. *The conjugacy problem is solvable in one-relator groups with torsion.*

Complete details, extensions, and applications of the results in this note will be submitted in a paper later.

I wish to acknowledge the encouragement given by Gilbert Baumslag.

BIBLIOGRAPHY

1. G. Baumslag, *Groups with one defining relator*, J. Austral. Math. Soc. 4 (1964), 385–392.

2. G. Baumslag and K. W. Gruenberg, *Some reflections on cohomological dimension and freeness*, J. Algebra **6** (1967), 394–409.

3. M. Greendlinger, *Dehn's algorithm for the word problem*, Comm. Pure Appl. Math. **13** (1960), 67–83.

4. ———, *On Dehn's algorithm for the conjugacy and word problems with applications*, Comm. Pure Appl. Math. **13** (1960), 641–677.

5. ———, *Solution by means of Dehn's generalized algorithm of the conjugacy problem for a class of groups which coincide with their anti-centers*, Dokl. Akad. Nauk SSSR **158** (1964), 1254–1256 = Soviet Math. Dokl. **5** (1964), 1384–1386.

6. R. C. Lyndon, *Cohomology theory of group extensions*, Duke Math. J. **15** (1948), 271–292.

7. ———, *Cohomology theory of groups with a single defining relation*, Ann. of Math. **52** (1950), 650–665.

8. ———, *Dependence and independence in free groups*, J. Reine Angew. Math. **210** (1962), 148–174.

9. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*, Interscience, New York, 1966.

10. N. S. Mendelsohn and Rimhak Ree, *Free subgroups of groups with a single defining relation*, Notices Amer. Math. Soc. **14** (1967), 90.

11. V. V. Soldatova, *On a class of finitely presented groups*, Dokl. Akad. Nauk SSSR **172** (1967), 1276–1277 = Soviet Math. Dokl. **8** (1967), 279–280.

Fairleigh Dickinson University and
University College of Townsville, Queensland, Australia